The Mathematics of Game Shows

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These are the course notes for a class on The Mathematics of Game Shows which I taught at the University of South Carolina (through their Honors College) in Fall 2016, and again in Spring 2018. They are in the middle of revisions, being made as I teach the class a second time.

Click here for the course website and syllabus:

Link: The Mathematics of Game Shows – Course Website and Syllabus

I welcome feedback from anyone who reads this (please e-mail me at `thorne[at]math.sc.edu`).

The notes contain clickable internet links to clips from various game shows, hosted on the video sharing site Youtube (`www.youtube.com`). These materials are (presumably) all copyrighted, and as such they are subject to deletion. I have no control over this. Sorry! If you encounter a dead link I recommend searching Youtube for similar videos. The Price Is Right videos in particular appear to be ubiquitous.

I would like to thank Bill Butterworth, Paul Dreyer, and **all** of my students for helpful feedback. I hope you enjoy reading these notes as much as I enjoyed writing them!

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1 Introduction

To begin, let's watch some game show clips and investigate the math behind them.

Here is a clip from the game show **Deal or No Deal**:

Link: Deal Or No Deal – Full Episode

(If you are reading this on a computer with an internet connection, clicking on any line labeled 'Link' should bring up a video on a web browser.)

> **Game Description (Deal or No Deal):** A contestant is presented with 26 briefcases, each of which contains some amount of money from \$0.01 to \$1,000,000; the amounts total \$3,418,416.01, and average \$131477.53. The highest prizes are \$500,000, \$750,000, and \$1,000,000.
>
> The contestant chooses one briefcase and sets it aside. That is the briefcase she is playing for. Then, one at a time, she is given the opportunity the opportunity to open other briefcases and see what they contain. This narrows down the possibilities for the selected briefcase.
>
> Periodically, the 'bank' offers to buy the contestant out, and proposes a 'deal': a fixed amount of money to quit playing. The contestant either accepts one of these offers, or keeps saying 'no deal' and (after opening all the other briefcases) wins the money in her original briefcase.

The **expected value** of a game is the average amount of money you expect to win. (We'll have much more to say about this.) So, at the beginning of the game, the expected value of the game is \$131,477.53, presuming the contestant rejects all the deals. In theory, that means that the contestant should be equally happy to play the game or to receive \$131,477.53. (Of course, this may not be true in practice.)

Now, consider this clip after the contestant has chosen six of the briefcases. Losing the \$500,000 was painful, but the others all had small amounts. After six eliminations, the total amount of prize money remaining is \$2,887,961.01, and the average is \$144,398.05 – higher than it was before. The banker offers him \$40,000 to stop playing. Since that is much lower than his expected value, understandably he refuses the offer and continues to play.

We now turn to the first game from this clip of The Price Is Right:

Link: The Price Is Right - Full Episode

> **Game Description (Contestants' Row - The Price Is Right):** Four contestants are shown an item up for bid. In order, each guesses its price (in whole dollars). You can't use a guess that a previous contestant used. The winner is the contestant who bids the closest to the actual price without going over.

In the clip, the contestants are shown some scuba equipment, and they bid 750, 875, 500, and 900 in that order. The actual price is \$994, and the fourth contestant wins. What can we say about the contestants' strategy?

Who bid wisely? We begin by describing the results of the bidding. Let n be the price of the scuba gear.

- The first contestant wins if $750 \leq n \leq 874$.
- The second contestant wins if $875 \leq n \leq 899$.
- The third contestant wins if $500 \leq n \leq 749$.
- The fourth contestant wins if $900 \leq n$.
- If $n < 500$, then the bids are all cleared and the contestants start over.

We can see who did well before we learn how much the scuba gear costs. Clearly, the fourth contestant did well. If the gear is worth anything more than \$900 (which is plausible), then she wins. The third contestant also did well: he is left with a large range of winning prices – 250 of them to be precise. The second contestant didn't fare well at all: although his bid was close to the actual price, he is left with a very small winning range. This is typical for this game: it is a big disadvantage to go early.

The next question to ask is: could any of the contestants have done better?

We begin with the fourth contestant. Here the answer is *yes*, and her strategy is **dominated** by a bid of \$876, which would win whenever $900 \leq n$, and in addition when $876 \leq n \leq 899$. In other words: *a bid of \$876 would win every time a bid of \$900 would, but not vice versa*. Therefore it is *always* better to instead bid \$876.

Taking this analysis further, we see that there are exactly four bids that make sense: 876, 751, 501, or 1. Note that each of these bids, except for the one-dollar bid, screws over one of her competitors, and this is not an accident: Contestant's Row is a **zero-sum game** – if someone else wins, you lose. If you win, everyone else loses.

The analysis gets much more subtle if we look at the *third* contestant's options. **Assume that the fourth contestant will play optimally** (an assumption which is very often not true in practice). Suppose, for example, that the third contestant believes that the scuba gear costs around \$1000. The previous bids were \$750 and \$875. Should he follow the same reasoning and bid \$876? Maybe, but this exposes him to a devastating bid of \$877.

There is much more to say here, but we go on to a different example.

Game Description (Jeopardy, Final Round): Three contestants start with a variable amount of money (which they earned in the previous two rounds). They are shown a category, and are asked how much they wish to wager on the final round. The contestants make their wagers privately and independently.

After they make their wagers, the contestants are asked a trivia question. Anyone answering correctly gains the amount of their wager; anyone answering incorrectly loses it.

Link: Final Jeopardy – Shakespeare

Perhaps here an English class would be more useful than a math class! This game is difficult to analyze; unlike our two previous examples, the players play *simultaneously* rather than *sequentially*.

In this clip, the contestants start off with \$9,400, \$23,000, and \$11,200 respectively. It transpires that nobody knew who said that *the funeral baked meats did coldly furnish forth the marriage tables*. (Richard III? Really? When in doubt, guess Hamlet.) The contestants bid respectively \$1801, \$215, and \$7601.

We will save further analysis for later, but we will make one note now: the second contestant can obviously win. If his bid is less than \$600, then even if his guess is wrong he will end up with more than \$22,400.

In the meantime, imagine that the contestants started with \$6,000, \$8,000, and \$10,000. Then the correct strategy becomes harder to determine.

2 Probability

2.1 Sample Spaces and Events

At the foundation of any discussion of game show strategies is a discussion of *probability*. You have already seen this informally, and we will work with this notion somewhat more formally.

Definition 1 (Sample spaces and events): A **sample space** is the set of all possible outcomes of a some process. An **event** is any subset of the sample space.

Example 2: You roll a die. The sample space consists of all numbers between one and six.

Using formal mathematical notation, we can write

$$S = \{1, 2, 3, 4, 5, 6\}.$$

We can use the notation $\{\dots\}$ to describe a set and we simply list the elements in it.

Let E be the *event* that you roll an even number. Then we can write

$$E = \{2, 4, 6\}.$$

Alternatively, we can write

$$E = \{x \in S \ : \ x \text{ is even}\}.$$

Both of these are correct.

Example 3: You choose at random a card from a poker deck. The sample space is the set of all 52 cards in the deck. We could write it

$$\begin{aligned} S = \{&A\clubsuit, K\clubsuit, Q\clubsuit, J\clubsuit, 10\clubsuit, 9\clubsuit, 8\clubsuit, 7\clubsuit, 6\clubsuit, 5\clubsuit, 4\clubsuit, 3\clubsuit, 2\clubsuit, \\ &A\diamondsuit, K\diamondsuit, Q\diamondsuit, J\diamondsuit, 10\diamondsuit, 9\diamondsuit, 8\diamondsuit, 7\diamondsuit, 6\diamondsuit, 5\diamondsuit, 4\diamondsuit, 3\diamondsuit, 2\diamondsuit, \\ &A\heartsuit, K\heartsuit, Q\heartsuit, J\heartsuit, 10\heartsuit, 9\heartsuit, 8\heartsuit, 7\heartsuit, 6\heartsuit, 5\heartsuit, 4\heartsuit, 3\heartsuit, 2\heartsuit, \\ &A\spadesuit, K\spadesuit, Q\spadesuit, J\spadesuit, 10\spadesuit, 9\spadesuit, 8\spadesuit, 7\spadesuit, 6\spadesuit, 5\spadesuit, 4\spadesuit, 3\spadesuit, 2\spadesuit\} \end{aligned}$$

but writing all of that out is annoying. An English description is probably better.

Example 4: You choose two cards at random from a poker deck. Then the sample space is the set of all pairs of cards in the deck. For example, $A\spadesuit A\heartsuit$ and $7\clubsuit 2\diamondsuit$ are elements of this sample space,

This is definitely too long to write out every element, so here an English description is probably better. (There are exactly 1,326 elements in this sample space.) Some events are easier to describe – for example, the event that you get a pair of aces can be written

$$E = \{A\spadesuit A\heartsuit, A\spadesuit A\diamondsuit, A\spadesuit A\clubsuit, A\heartsuit A\diamondsuit, A\heartsuit A\clubsuit, A\clubsuit A\diamondsuit\}$$

and has six elements. If you are playing Texas Hold'em, your odds of being dealt a pair of aces is exactly $\frac{6}{1326} = \frac{1}{221}$, or a little under half a percent.

Let's look at a simple example from the Price Is Right – the game of **Squeeze Play**:

Link: The Price Is Right - Squeeze Play

Game Description (Squeeze Play (The Price Is Right)): You are shown a prize, and a five- or six-digit number. The price of the prize is this number with one of the digits removed, other than the first or the last.

The contestant is asked to remove one digit. If the remaining number is the correct price, the contestant wins the prize.

In this clip the contestant is shown the number 114032. Can we describe the game in terms of a sample space?

It is imporrtant to recognize that **this question is not precisely defined. Your answer will depend on your interpretation of the question!** This is probably very much *not* what you are used to from a math class.

Here's one possible interpretation. Either the contestant wins or loses, so we can describe the sample space as

$$S = \{\text{you win}, \text{you lose}\}.$$

Logically there is nothing wrong with this. But it doesn't tell us very much about the structure of the game, does it?

Here is an answer I like better. We write

$$S = \{14032, 11032, 11432, 11402\},$$

where we've written 14032 as shorthand for 'the price of the prize is 14032'.

Another correct answer is

$$S = \{2, 3, 4, 5\},$$

where here 2 is shorthand for 'the price of the prize has the second digit removed.'

Still another correct answer is

$$S = \{1, 4, 0, 3\},$$

where here 1 is shorthand for 'the price of the prize has the 1 removed.'

All of these answers make sense, and all of them require an accompanying explanation to understand what they mean.

The contestant chooses to have the 0 removed. So the event that the contestant wins can be described as $E = \{11432\}$, $E = \{4\}$, or $E = \{0\}$, depending on which way you wrote the sample space. (Don't mix and match! Once you choose how to write your sample space, you need to describe your events in the same way.) If all the possibilities are equally likely, the contestant has a one in four chance of winning.

The contest guesses correctly and is on his way to Patagonia!

Definition 5 ($N(S)$): If S is any set (for example a sample space or an event), write $N(S)$ for the number of elements in it.

In this course we will always assume this number is *finite*.

Definition 6 (Probability): Suppose S is a sample space, **in which we assume that all outcomes are equally likely**.

For each event E in S, the **probability of E, denoted $P(E)$**, is

$$P(E) = \frac{N(E)}{N(S)}.$$

Example 7: You roll a die, so $S = \{1, 2, 3, 4, 5, 6\}$.

1. Let E be the event that you roll a 4, i.e., $E = \{4\}$. Then $P(E) = \frac{1}{6}$.
2. Let E be the event that you roll an odd number, i.e., $E = \{1, 3, 5\}$. Then $P(E) = \frac{3}{6} = \frac{1}{2}$.

Example 8: You draw one card from a deck, with S as before.

1. Let E be the event that you draw a spade. Then $N(E) = 13$ and $P(E) = \frac{13}{52} = \frac{1}{4}$.
2. Let E be the event that you draw an ace. Then $N(E) = 4$ and $P(E) = \frac{4}{52} = \frac{1}{13}$.

3. Let E be the event that you draw an ace or a spade. What is $N(E)$? There are thirteen spades in the deck, and there are three aces which are not spades. Don't double count the ace of spades!

   So $N(E) = 13 + 3 = 16$ and $P(E) = \frac{16}{52} = \frac{4}{13}$.

Example 9: In a game of Texas Hold'em, you are dealt two cards at random in first position. You decide to raise if you are dealt a pair of sixes or higher, ace-king, or ace-queen, and to fold otherwise.

The sample space has 1326 elements in it. The event of two-card hands which you are willing to raise has 86 elements in it. (If you like, write them all out. Later we will discuss how this number can be computed more efficiently!)

Since all two card hands are equally likely, the probability that you raise is $\frac{86}{1326}$, or around one in fifteen.

Now, here is an important example:

Warning Example 10: You roll two dice and sum the totals. What is the probability that you roll a 7?

The result can be anywhere from 2 to 12, so we have

$$S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

and $E = \{7\}$. Therefore, we might be led to conclude that $P(E) = \frac{N(E)}{N(S)} = \frac{1}{11}$.

Here is another solution. We can roll anything from 1 to 6 on the first die, and the same for the second die, so we have

$$S = \{11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36,\\ 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66\}.$$

We list all the possibilities that add to 7:

$$E = \{16, 25, 34, 43, 52, 61\}$$

And so $P(E) = \frac{6}{36} = \frac{1}{6}$.

We solved this problem two different ways and got two different answers. **This illustrates the importance of our assumption that every outcome in a sample space will be equally likely**. This might or not be true in any particular situation. And one can't tell just from knowing what E and S are – one has to understand the actual situation that they are modelling.

We know that a die (if it is equally weighted) is equally likely to come up 1, 2, 3, 4, 5, or 6. So we can see that, according to our second interpretation, all the possibilities are still equally likely because all combinations are explicitly listed. But there is no reason why all the sums should be equally likely.

For example, consider the trip to Patagonia. If we assume that all outcomes are equally likely, the contestant's guess has a 1 in 4 chance of winning. But the contestant correctly guessed that over $14,000 was implausibly expensive, and around $11,000 was more reasonable.

Often, all events are *approximately* equally likely, and considering them to be exactly equally likely is a useful simplifying assumption.

We now take up the game **Rat Race** from The Price Is Right. (We will return to this example again later.)

Link: The Price Is Right - Rat Race

Game Description (Rat Race (The Price Is Right)): The game is played for three prizes: a small prize, a medium prize, and a car.

There is a track with five wind-up rats (pink, yellow, blue, orange, and green). They will be set off on a race, where they will finish in (presumably) random order.

The contestant has the opportunity to pick up to three of the rats: she guesses the price of three small items, and chooses one rat for each successful attempt.

After the rats race, she wins prizes if one or more of her rats finish in the top three. If she picked the third place rat, she wins the small prize; if she picked the second place rat, she wins the medium prize; if he picked the first place rat, she wins the car. (Note that it is possible to win two or even all three prizes.)

Note that except for knowing the prices of the small items, there is no strategy. The rats are (we presume) equally likely to finish in any order.

In this example, the contestant correctly prices two of the items and picks the pink and orange rats.

Problem 1. *Compute the probability that she wins the car.*

Solution 1. Here's the painful solution: describe all possible orderings in which the rats could finish. We can describe the sample space as

$$S = \{POB, POR, POG, PBR, PBG, PRG, \dots, \dots\}$$

where the letters indicate the ordering of the first three rats to finish. Any such ordering is equally likely. The sample space has sixty elements, and if you list them all you will see that exactly twenty-four of them start with P or G. So the probability is $\frac{24}{60} = \frac{2}{5}$.

Solution 2. Do you see the easier solution? To answer the problem we were asked, we only care about the **first** rat. So let's ignore the second and third finishers, and write the sample space as

$$S = \{P, O, B, R, G\}.$$

The event that she wins is

$$E = \{P, G\},$$

and so $P(E) = \frac{N(E)}{N(S)} = \frac{2}{5}$.

Solution 3 (Wrong). Here's another possible solution, which turns out to be wrong. It doesn't model the problem well, and it's very instructive to understand why.

As the sample space, take all combinations of one rat and which order it finishes in:

$$\begin{aligned} S = \{&\text{Pink rat finishes first,} \\ &\text{Pink rat finishes second,} \\ &\text{Pink rat finishes third,} \\ &\text{Pink rat finishes fourth,} \\ &\text{Pink rat finishes fifth,} \\ &\text{Yellow rat finishes first,} \\ &\text{etc.}\} \end{aligned}$$

This sample space indeed lists a lot of different things that could happen. But how would you describe the event that the contestant wins? If the pink or orange rat finishes first, certainly she wins. But what if the yellow rat finishes third? Then maybe she wins, maybe she loses. There are several problems with this sample space:

- The events are not mutually exclusive. It can happen that **both** the pink rat finishes second, **and** the yellow rat finishes first. A sample space should be described so that **exactly one of the outcomes will occur**.

  Of course, a meteor could strike the television studio, and Drew, the contestant, the audience, and all five rats could explode in a giant fireball. But we're building *mathematical models* here, and so we can afford to ignore remote possibilities like this.

- In addition, you can't describe the event 'the contestant wins' as a subset of the sample space. What if the pink rat finishes fifth? The contestant also has the orange rat. It is ambigious whether this possibility should be part of the event or not.

> **Advice:** **Note that it is a very good thing to come up with wrong ideas** – provided that one then examines them critically, realizes that they won't work, and rejects them. Indeed, very often when solving a problem, your first idea will often be incorrect. Welcome this process – it is where the best learning happens.

This also means that you are not truly finished with a problem when you write down an answer. You are only finished when you think about your answer, check your work (if applicable), and make sure that your answer makes sense.

Solution 4. The contestant picked two rats, and we may list the positions in which they finish. For example, write 25 if her rats came in second and fifth. Then we have

$$S = \{12, 13, 14, 15, 23, 24, 25, 34, 35, 45\},$$

and the event that she wins is described by

$$E = \{12, 13, 14, 15\}$$

with

$$P(E) = \frac{N(E)}{N(S)} = \frac{4}{10} = \frac{2}{5}.$$

Solution 5. We list the positions in which the pink and orange rats finish, in that order. Here we have

$$S = \{12, 21, 13, 31, 14, 41, 15, 51, 23, 32, 24, 42, 25, 52, 35, 53, 45, 54\}$$

and

$$E = \{12, 21, 13, 31, 14, 41, 15, 51\},$$

with

$$P(E) = \frac{N(E)}{N(S)} = \frac{8}{20} = \frac{2}{5}.$$

Yet another correct solution!

Although one solution is enough, it is good to come up with multiple solutions. For one thing, it helps us understand the problem better. Beyond that, different solutions might generalize in different directions. For example, Solution 4 tells us all the information that the contestant might care about, and is a good sample space for analyzing other problems as well.

Problem 2. *Compute the probability that she wins both the car and the meal delivery.*

We could use the sample space given in Solution 4 above. We will instead present an alternate solution here:

Here we care about the first *two* rats. We write

$$S = \{PO, PB, PR, PG, OP, OB, OR, OG, BP, BO, BR, BG, RP, RO, RB, RG, GP, GO, GB, GR\}.$$

The sample space has twenty elements in it. ($20 = 5 \times 4$: there are 5 possibilities for the first place finisher, and (once we know who wins) 4 for the second. More on this later.) The event that she wins is

$$\{PO, OP\},$$

since her two rats have to finish in the top two places – but either of them can finish first. We have $P(E) = \frac{N(E)}{N(S)} = \frac{2}{20} = \frac{1}{10}$.

Problem 3. *Compute the probability that she wins all three prizes.*

Zero. Duh. She only won two rats! Sorry.

2.2 The Addition and Multiplication Rules

Working out these examples – and especially the Rat Race example – should give you the intuition that there is mathematical structure intrinsic to these probability computations. We will single out two rules that are particularly useful in solving problems.

Theorem 11 (The Addition Rule for Probability): Suppose E and F are two *disjoint* events in the *same sample space* – i.e., they don't overlap. Then

$$P(E \text{ or } F) = P(E) + P(F).$$

The addition rule is an example of a *mathematical theorem* – a general mathematical statement that is always true. In a more abstract mathematics course, we might *prove* each theorem we state. Here, we will often informally explain why theorems are true, but it is not our goal to offer formal proofs.

Example 12: You roll a die. Compute the probability that you roll either a 1, or a four or higher.

Solution. Let $E = \{1\}$ be the event that you roll a 1, and $E = \{4, 5, 6\}$ be the event that you roll a 4 or higher. Then

$$P(E \text{ or } F) = P(E) + P(F) = \frac{1}{6} + \frac{3}{6} = \frac{4}{6} = \frac{2}{3}.$$

Example 13: You draw a poker card at random. What is the probability you draw either a heart, or a black card which is a ten or higher?

Solution. Let E be the event that you draw a heart. As before, $P(E) = \frac{13}{52}$.

Let F be the event that you draw a black card ten or higher, i.e.,

$$F = \{A\clubsuit, K\clubsuit, Q\clubsuit, J\clubsuit, 10\clubsuit, A\spadesuit, K\spadesuit, Q\spadesuit, J\spadesuit, 10\spadesuit\}.$$

Then $P(F) = \frac{10}{52}$.

So we have

$$P(E \text{ or } F) = \frac{13}{52} + \frac{10}{52} = \frac{23}{52}.$$

Example 14: You draw a poker card at random. What is the probability you draw either a heart, or a red card which is a ten or higher?

Solution. This doesn't have the same answer, because hearts are red. If we want to apply the addition rule, we have to do so carefully.

Let E be again the event that you draw a heart, with $P(E) = \frac{13}{52}$.

Now let F be the event that you draw a *diamond* which is ten or higher:

$$F = \{A\diamondsuit, K\diamondsuit, Q\diamondsuit, J\diamondsuit, 10\diamondsuit\}.$$

Now together E and F cover all the hearts and all the red cards at least ten, and there is no overlap. So we can use the addition rule.

$$P(E \text{ or } F) = P(E) + P(F) = \frac{13}{52} + \frac{5}{52} = \frac{18}{52}.$$

We won't state it formally as a theorem, but the additon rule also can be applied analogously with more than two events.

Example 15: Consider the Rat Race contestant from earlier. What is the probability that she wins any two of the prizes?

Solution 1. We will give a solution using the addition rule. (Later, we will give another solution using the Multiplication Rule.)

Recall that her chances of winning the car and the meal delivery were $\frac{1}{10}$. Let us call this event CM instead of E.

Now what are her chances of winning the car and the guitar? (Call this event CG.) Again $\frac{1}{10}$. If you like, you can work this question out in the same way. But it is best to observe that there is a natural symmetry in the problem. The rats are all alike and any ordering is equally likely. They don't know which prizes are in which lanes. So the probability has to be the same.

Finally, what is $P(MG)$, the probability that she wins the meal service and the guitar? Again $\frac{1}{10}$ for the same reason.

Finally, observe these events are all disjoint, because she can't possibly win more than two. So the probability is three times $\frac{1}{10}$, or $\frac{3}{10}$.

Here is a contrasting situation. Suppose the contestant had picked all three small prices correctly, and got to choose three of the rats. In this case, the probability she wins both the car and the meal service is $\frac{3}{10}$, rather than $\frac{1}{10}$. (You can either work out the details yourself, or else take my word for it.)

But this time the probability that she wins two prizes is *not* $\frac{3}{10} + \frac{3}{10} + \frac{3}{10}$, because now the events CM, CG, and MG are not disjoint: it is possible for her to win all three prizes, and if she does, then all of CM, CG, and MG occur!

It turns out that in this case the probability that she wins *at least* two is $\frac{7}{10}$, and the probability that she wins *exactly* two is $\frac{3}{5}$.

The Multiplication Rule. The multiplication rule computes the probability that two events E and F **both** occur. Here they are events in **different** sample spaces.

Theorem 16 (The Multiplication Rule): If E and F are events in different sample spaces, then we have

$$P(E \text{ and } F) = P(E) \times P(F).$$

Although this formula is not always valid, it *is* valid in either of the following circumstances:

- The events E and F are *independent*.
- The probability given for F assumes that the event E occurs (or vice versa).

Example 17: You flip a coin twice. What is the probability that you flip heads both times?

Solution. We can use the multiplication rule for this. The probability that you flip heads if you flip a coin once is $\frac{1}{2}$. Coin flips are independent: flipping heads the first time doesn't make it more or less likely that you will flip heads the second time. So we multiply the probabilities to get $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Alternatively, we can give a direct solution. Let

$$S = \{HH, HT, TH, TT\}$$

and

$$E = \{HH\}.$$

Since all outcomes are equally likely,

$$P(E) = \frac{N(E)}{N(S)} = \frac{1}{4}.$$

Like the addition rule, we can also use the multiplication rule for more than two events.

Example 18: You flip a coin twenty times. What is the probability that you flip heads every time?

Solution. If we use the multiplication rule, we see that the probability is

$$\frac{1}{2} \times \frac{1}{2} \times \cdots \times \frac{1}{2} = \frac{1}{2^{20}} = \frac{1}{1048576}.$$

This example will illustrate the second use of the Multiplication Rule.

Example 19: Consider the Rat Race example again (as it happened in the video). What is the probability that the contestant wins both the car and the meal service?

Solution. This is not hard to do directly, but we illustrate the use of the multiplication rule.

The probability that she wins the car is $\frac{2}{5}$, as it was before. So we need to now compute the probability that she wins the meal service, *given that she won the car*.

This time the sample space consists of *four* rats: we leave out whichever one won the car. The event is that her remaining one rat wins the meal service, and so the probability of this event is $\frac{1}{4}$.

By the multiplication rule, the total probability is

$$\frac{2}{5} \times \frac{1}{4} = \frac{1}{10}.$$

Example 20: What is the probability that the contestant wins the car, and the car only?

Solution. This is similar to before, so we will be brief. The probability that she wins the car is $\frac{2}{5}$; *given this*, the probability that her remaining rat loses is $\frac{1}{2}$. So the answer is

$$\frac{2}{5} \times \frac{1}{2} = \frac{1}{5}.$$

Example 21: Suppose a Rat Race contestant prices all three prizes correctly and has the opportunity to race three rats. What is the probability she wins all three prizes?

Solution. The probability she wins the car is $\frac{3}{5}$, as before: the sample space consists of the five rats, and the event that she wins consists of the three rats she chooses. (Her probability is $\frac{3}{5}$ no matter which rats she chooses, under our assumption that they finish in a random order.)

Now assume that she wins the first prize. Assuming this, the probability that she wins the meals is $\frac{2}{4} = \frac{1}{2}$. The sample space consists of the four rats *other than the first place finisher,* and the event that she wins the meals consists of the two rats *other than the first place finishers.*

Now assume that she wins the first and second prizes. The probability she wins the guitar is $\frac{1}{3}$: the sample space consists of the three rats *other than the first two finishers,* and the event that she wins the meals consists of the single rat *other than the first two finishers.*

There is some subtlety going on here! To illustrate this, consider the following:

Example 22: Suppose a Rat Race contestant prices all three prizes correctly and has the opportunity to race three rats. What is the probability she wins the meal service?

Solution. There are five rats in the sample space, she chooses three of them, and each of them is equally likely to finish second. So her probability is $\frac{3}{5}$ (same as her probability of winning the car).

But didn't we just compute that her odds of winning the car are $\frac{1}{2}$? What we're seeing is something we'll investigate much more later. This probability $\frac{1}{2}$ is a **conditional** probability: it assumes that one of the rats finished first, and illustrates what is hopefully intuitive: if she wins first place with one of her three rats, she is less likely to also win second place.

Let's see an incorrect application of the multiplication rule along these lines:

Warning Example 23: Suppose we compute again the probability that she wins all three prizes with three rats. She has a $\frac{3}{5}$ probability of winning first, a $\frac{3}{5}$ probability of winning second, and a $\frac{3}{5}$ probability of winning third. By the multiplication rule, the probability that all of these events occur is

$$\frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{27}{125}.$$

What is wrong with this reasoning is that these events are *not independent.* Once one of her rats wins, she only has two remaining rats (out of four) to win the other two prizes, and so these probabilities must be recalculated.

In the previous examples, it would have been relatively simple to avoid using the multiplication rule, and instead to write out an entire sample space as appropriate. Here is an example that would be very time-consuming to do that way, but is easy using the multiplication rule:

Example 24: You draw two cards at random from a poker deck. What is the probability that you get two aces?

Solution. The probability that the first card is an ace is $\frac{4}{52}$ or $\frac{1}{13}$: there are 52 cards, and 4 of them are aces.

We now compute the probability that the second card is an ace, *given that the first card was an ace*. There are now 51 cards left in the deck, and only 3 of them are aces. So this probability is $\frac{3}{51} = \frac{1}{17}$.

So the probability that both cards are aces is

$$\frac{1}{13} \times \frac{1}{17} = \frac{1}{221}.$$

Here is a poker example. A poker hand consists of five cards, and it is a **flush** if they are all of the same suit.

Example 25: You draw five cards at random from a poker deck. What is the probability that you draw a flush?

Solution. The most straightforward solution (no shortcuts) uses both the addition and multiplication rules. We first compute the probability of drawing five *spades*, in each case presuming the previous cards were all spades. By the multiplication rule, this is

$$\frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{9}{48} = \frac{33}{66640} \approx 0.0005.$$

By symmetry, the same is true of the probability of drawing five hearts, or five diamonds, or five clubs. Since these events are all mutually exclusive, we can add them to get

$$4 \cdot 0.0005 = 0.002,$$

or a roughly 1 in 500 chance.

Shortcut. The above solution is completely correct. Here is an optional shortcut.

We are happy with the first card, no matter what it is. The probability that the second card is of the same suit is then $\frac{12}{51}$, and the probability that the third matches the first two is $\frac{11}{50}$, and so on. So the total probability is

$$\frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{9}{48} = \frac{33}{16660} \approx 0.002.$$

We computed the probability of all suits simultaneously. We didn't multiply by $\frac{13}{52} = \frac{1}{4}$ at the beginning, and we didn't multiply by 4 at the end.

In general, it is not very important to be able to find such shortcuts. The first solution is, after all, completely correct.

However, it is highly recommended that, after you find one solution, you read others. Understanding how different solution methods can lead to the same correct answer is highly valuable for building your intuition.

Warning Example 26: Try to use the same solution method to compute the odds of being dealt a **straight**: five cards of consecutive denominations (and any suits).

You won't get anywhere. (Try it!) We'll need to develop our tools further.

Press Your Luck. Here is a bit of game show history. The following clip comes from the game show Press Your Luck on May 19, 1984.

Link: Press Your Luck – Michael Larson

The rules are complicated, and we won't explain them fully. But in summary, as long as contestants have any remaining spins, they have the opportunity to 'press their luck' (keep spinning) or pass their spins to another player. (You may use or pass any spins that you have 'earned', but if they are passed to you then you must take them.) Contestants keep any money or prizes that they earn, but the cartoon characters are 'Whammies' and if you land on one then you lose everything.

Here Michael Larsen smashed the all-time record by winning $110,237. The truly fascinating clip starts at around 17:00, where Larson continues to press his luck, to the host's increasing disbelief. Over and over and over again, Larson not only avoided the whammies but continued to hit spaces that allowed for an extra spin.

Example 27: What is the probability of this happening?

Solution. This is an exercise not only in probability computations, but also in observation and modeling. The question is not totally precise, and we have to make it precise before we can answer it. Moreover, we will have to introduce some simplifying assumptions before we can take a decent crack at it. All of this can be done in different ways, and **this is one of multiple possible answers**. (Indeed, during class, some students presented answers which were more thorough than this!)

If your logic is sound then you should get something *roughly* similar – this is what it is like to do math 'in the real world'!

Watching the video, we see that on 28 consecutive spins, Larson avoided all the whammies and hit a space that afforded him an extra spin. We will ask the probability of *that*.

However, the configuration of the board keeps changing! Whammies, money, and extra spins keep popping in and out. We may observe that **on average** there are approximately five spaces worth money and an extra spin. Since there are 18 spaces, we will **assume** that the probability of landing on a 'good space' is $\frac{5}{18}$. This is probably not exactly true, but it is at least approximately true.

With the modeling done, the math is easy. All the spins are independent, and the probability of Larson pulling off such a feat is

$$\left(\frac{5}{18}\right)^{28} \approx 0.000000000000026\%.$$

If you see such a low probability, but the event actually happened, you should question your assumptions. Here our most fundamental assumption is that the process is random. In truth, as you may have guessed, there are patterns in the board's movement. Larson had taped previous episodes of this show, painstakingly watched them one frame at a time, and figured out what the patterns were. In short, he had cheated.

Warning Example 28: This business of interpreting real-world data in different ways *can* be taken too far, *especially* if one cherry-picks the data with an eye towards obtaining a desired conclusion.

This is spectacularly illustrated by the following famous research paper:

Link: Neural correlates of interspecies perspective taking in the post-mortem Atlantic Salmon

Here was the task successfully performed by the salmon:

> The salmon was shown a series of photographs depicting human individuals in social situations with a specified emotional valence. The salmon was asked to determine what emotion the individual in the photo must have been experiencing.

An impressive feat, especially considering:

> The salmon was approximately 18 inches long, weighed 3.8 lbs, and was not alive at the time of scanning.

Card Sharks. You might be interested[1] in the following clip of the game Card Sharks:

[1] This game was treated more extensively in a previous version of the notes, but the computations were rather messy, and so are mostly left out here.

Link: Card Sharks

At each stage of the game, you can figure out the probability that you can successfully guess whether the next card will be higher or lower. You can thus deduce the winning strategy.

Theoretically the game is not too difficult, but in practice the computations can be very messy.

Ellen's Game of Games – Hot Hands. Ellen's Game of Games is a new game show, launched by host Ellen DeGeneres in 2017. Like The Price Is Right, the game involves lots of mini-games.

Here are two consecutive playings of *Hot Hands*:

Link: Ellen's Game of Games – Hot Hands

I could not find accurate rules for the game listed on the Internet (the Wikipedia page is conspicuously wrong); perhaps, they change at DeGeneres' whim. Roughly, they seem to be as follows:

> **Game Description (Ellen's Game of Games – Hot Hands):** The contestant is shown photos of celebrities, and has 30 seconds to identify as celebrities as she can, with at most three seconds for each one. She wins an amount of money that increases with the number of correct guesses.

Some things are left ambiguous. For example, suppose the contestant *immediately* passed on any celebrity she didn't *immediately* recognize; would she have to wait three seconds, and would she be shown arbitrarily many celebrities?

As actually played, the game is hard to analyze mathematically, but we can analyze the following oversimplification of the game: *assume that the contestant has the opportunity to identify exactly ten celebrities.*

The outcome is not random: either she knows the celebrity or she doesn't. If you ever get the chance to play this game, then these notes won't be as useful as a random back issue of *People* magazine. Nevertheless, we can ask a couple of questions:

> **Example 29:** If the contestant has the chance to guess at ten celebrities, and has a 50-50 chance at each, what is the chance of guessing them all correctly?
>
> **Solution.** Hopefully this is easy by now, the answer is
>
> $$\left(\frac{1}{2}\right)^{10} = \frac{1}{1024}.$$

Example 30: If the contestant has the chance to guess at ten celebrities, and has a 50-50 chance at each, what is the chance of guessing *at least nine* correctly?

Solution. By the same reasoning, the probability of any particular sequence of answers is $\frac{1}{1024}$. For example, the following sequence has probability $\frac{1}{1024}$: first question wrong, second question right, third question right, fourth question right, fifth question wrong, sixth question right, seventh question wrong, eighth question wrong, ninth question right, tenth question right.

We could have, independently, listed 'right' or 'wrong' after each of the question numbers. The point is that we made a particular choice, and no matter which particular choice we made the probability is the same.

So we want to use the addition rule, and add up all the different ways in which she could get at least nine questions correct:

- She could get all ten questions correct.
- She could get the first question wrong, and the remaining questions correct.
- She could get the second question wrong, and the remaining questions correct.
- There are eight more possibilities like this. Indeed, there are ten questions, and she could get any one of them wrong and answer all the remaining ones correctly.

So we have listed 11 distinct events – one, getting all the questions correct, and the remaining ten getting any one of the other 10 questions wrong and the others correct. Each has probability $\frac{1}{1024}$, so the total probability is $\frac{11}{1024}$.

Here is another question: what is the probability of getting exactly five questions right? The answer turns out to be $\frac{252}{1024}$: there are exactly 252 distinct ways in which she could get exactly five questions right. How to compute *that*?!

We could list them all out explicitly, but that sounds... a little bit tedious. We see that to be good at probability, we need to be good at counting. The art of counting, without actually counting, is called *combinatorics*. We develop a couple of principles in the next section, and we will return to this topic again in Chapter 4.

2.3 Permutations and Factorials

Here is a clip of the Price Is Right game **Ten Chances**:

Link: The Price Is Right – Ten Chances

Game Description (The Price Is Right – Ten Chances): The game is played for a small prize, a medium prize, and a car, in that order.

The small prize has two (distinct) digits in its price, and the contestant is shown three digits – the two digits in the prize of the prize, plus an extra one. She must guess the price of the small prize by using two of these digits in some order. On a correct guess she wins it and moves on to the next prize.

She then must guess the price of the medium prize: it has three (distinct) digits in its price, and she is shown four.

Finally, if she wins the medium prize she gets to play for the car: it has five digits in its price, and this time she is shown all five digits without any decoy.

She has ten chances, total, to win as many of the prizes as she can.

So, for example, for the car the contestant is shown the digits 6, 8, 1, 0, and 7. Her first guess is $18,670 – sounds reasonable enough, but it's wrong.

Example 31: In the clip, the price of the small prize (a popcorn maker) contains two digits from $\{4, 0, 5\}$. If all possibilities are equally likely to be correct, what are her odds of guessing correctly on the first try?

Solution 1. The sample space of all possibilities is

$$\{04, 05, 40, 45, 50, 54\}.$$

The contestant guesses 45, but in any case we hypothesized that each was equally likely to occur, so her odds of winning are $\frac{1}{6}$.

Solution 2. We use the multiplication rule. There are three different possibilities for the first digit, and exactly one of them is correct. The probability that she gets the first digit correct is therefore $\frac{1}{3}$.

If she got the first digit correct, then there are two remaining digits, and the probability that she picks the correct one is $\frac{1}{2}$.

Thus the probability of getting both correct is $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$.

Note also that unless she does something particularly dumb, she is guaranteed to have at least four chances at the other two prizes.

You might notice, by the way, that our assumption that the possibilities are equally likely is unrealistic. Surely the popcorn maker's price is not 04 dollars? They're not that cheap, and even if they were, you'd write 4 and not 04.

Indeed:

Example 32: If the contestant had watched The Price Is Right a lot, she'd know that *the prices all end in zero.* If she uses this fact to her advantage, now what is her probability of guessing correctly on the first try?

Solution. The sample space gets shrunk to

$$\{40, 50\},$$

so she has a 1 in 2 chance.

For the karaoke machine, she chooses three digits from $\{2, 9, 0, 7\}$. One can compute similarly that her probability of guessing right on any particular try is $\frac{1}{24}$ (or $\frac{1}{6}$ if you know the last digit is zero).

Finally, the price of the car has the digits $\{6, 8, 1, 0, 7\}$ and this time she uses all of them. The sample space is too long to effectively write out. So we work out the analogue of Solution 2 above: Her odds of guessing the first digit are $\frac{1}{5}$. If she does so, her odds of guessing the second digit is $\frac{1}{4}$ (since she has used one up). If both these digits are correct, her odds of guessing the third digit is $\frac{1}{3}$. If these three are correct, her odds of guessing the fourth digit are $\frac{1}{2}$. Finally, **if** the first four guesses are correct then the last digit is automatically correct by process of elimination. So the probability she wins is

$$\frac{1}{5} \times \frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} \times 1 = \frac{1}{120}.$$

Here the number 120 is equal to 5!, or 5 **factorial**. In math, an exclamation point is read 'factorial' and it means the product of all the numbers up to that point. We have

$$\begin{aligned}
1! &= 1 && = 1 \\
2! &= 1 \times 2 && = 2 \\
3! &= 1 \times 2 \times 3 && = 6 \\
4! &= 1 \times 2 \times 3 \times 4 && = 24 \\
5! &= 1 \times 2 \times 3 \times 4 \times 5 && = 120 \\
6! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 && = 720 \\
7! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 && = 5040 \\
8! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 && = 40320 \\
9! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 && = 362880 \\
10! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 && = 3628800,
\end{aligned}$$

and so on. We also write $0! = 1$. Why 1 and not zero? 0! means 'don't multiply anything', and we think of 1 as the starting point for multiplication. (It is the *multiplicative identity*, satisfying $1 \times x = x$ for all x.) So when we compute 0! it means we didn't leave the starting point.

These numbers occur **very** commonly in the sorts of questions we have been considering, for reasons we will shortly see.

Example 33: The contestant wins the first two prizes in seven chances, and has three chances left over. If each possibility for the price of the car is equally likely, then what is the probability that she wins it?

Solution. The answer is three divided by $N(S)$, the number of elements in the sample space. So if we could just compute $N(S)$, we'd be done.

Here there is a trick! She guesses 18670, and we know that the probability that this is correct is $\frac{1}{N(S)}$: one divided by the number of total possible guesses. But we already computed the probability: it's $\frac{1}{120}$. Therefore, we know that $N(S)$ is 120, without actually writing it all out!

We just solved our first combinatorics problem: we figured out that there were 120 ways to rearrange the numbers $6, 8, 1, 0, 7$ without actually listing all the possibilities. We now formalize this principle.

Definition 34 (Strings and permutations): A **string** is any sequence of numbers, letters, or other symbols. For example, 01568 and 22045 are strings of numbers, ABC and xyz are strings of letters. Order matters: 01568 is not the same string as 05186.

A **permutation** of a string T is any reordering of T.

So, for example, if T is the string 1224, then 2124, 4122, 1224, and 2142 are all permutations of T. Note we *do* consider T itself to be a permutation of T, for the same reason that we consider 0 a number. It is called the **trivial permutation**.

We have the following:

Theorem 35 (Counting permutations): Let T be a string with n symbols, all of which are distinct. Then there are exactly $n!$ distinct permutations of T.

As with our earlier theorems, the hypothesis is necessary: the string 1224 has a repeated symbol, and there are not $4! = 24$ permutations of it, but in fact only 12 of them. We will see later how to count permutations when the strings have repeated symbols.

Why is the theorem true? Think about how to arrange the strings: there are n possibilities for the first symbol (all different, because the symbols are all distinct), $n - 1$ for the second, $n - 2$, and so on. This is essentially like the multiplication rule, only for counting instead of probability.

We now return to our Ten Chances contestant. Recall that, after the two small prizes, she has three chances to win the car.

Example 36: Suppose that the contestant has watched The Price Is Right a lot and so knows that *the last digit is the zero.* Compute the probability that she wins the car, given three chances.

Solution. Here her possible guesses consist of permutations of the string 6817, followed by a zero. There are $4! = 24$ of them, so her winning probability is $\frac{3}{24} = \frac{1}{8}$.

Her winning probability went up by a factor of exactly 5 – corresponding to the fact that $\frac{1}{5}$ of the permutations of 68107 have the zero in the last digit. Equivalently, a random permutation of 68107 has probability $\frac{1}{5}$ of having the zero as the last digit.

This is still not optimal. For example, suppose the contestant had guessed 81670. To any reader that considers that a likely price of a *Ford Fiesta*... I have a car to sell you.

Example 37: Suppose that the contestant knows that the *the last digit is the zero* and *the first digit is the one.* Compute the probability that she wins the car, given three chances.

Solution. Her guesses now consist of permutations of the string 867, with a 1 in front and followed by a zero. There are $3! = 6$ of them. She has three chances, so her chance of winning is $\frac{3}{6} = \frac{1}{2}$.

Note that it is only true of Ten Chances that car prices always end in zero – not of The Price Is Right in general. Here is a contestant who is very excited until she realizes the odds she is against:

Link: The Price Is Right – Three Strikes for a Ferrari

Game Description (The Price Is Right – Three Strikes): The game is played for a car; the price will usually be five digits but may occasionally be six. All the digits in the price of the car will be distinct.

Five (or six) tiles, one for each digit the price, are mixed in a bag with three 'strike' tiles. Each turn, she draws one of the tiles. If it is a strike, it is removed from play. If it is a digit, she has the opportunity to guess its position in the price of the car. If she guesses correctly, her guess is illuminated on the scoreboard and the tile is removed from play. If she guesses incorrectly, then the tile is returned to the bag.

She continues drawing and guessing until either (1) she has correctly identified the positions of all the digits in the price, in which case she wins the car; or (2) she draws all three strikes, in which case she loses.

2.4 Exercises

Most of these should be relatively straightforward, but there are a couple of quite difficult exercises mixed in here for good measure.

Starred exercises indicate optional exercises.

1. Card questions. In each question, you choose at random a card from an ordinary deck. What is the probability you –
   (a) Draw a spade?
   (b) Draw an ace?
   (c) Draw a face card? (a jack, queen, king, or an ace)
   (d) Draw a spade or a card below five?
2. Dice questions:
   (a) You roll two dice and sum the total. What is the probability you roll exactly a five? At least a ten? **Solution.** The sample space consists of 36 possibilities, 11 through 66. The first event can be described as $\{14, 23, 32, 41\}$ and has probability $\frac{4}{36} = \frac{1}{9}$. The second can be described as $\{46, 55, 64, 56, 65, 66\}$ and has probability $\frac{6}{36} = \frac{1}{6}$.
   (b) You roll three dice and sum the total. What is the probability you roll at least a 14? (This question is kind of annoying if you do it by brute force. Can you be systematic?) **Solution.** There are several useful shortcuts. Here is a different way than presented in lecture. The sample space consists of $6 \times 6 \times 6 = 216$ elements, 111 through 666. The event of rolling at least a 14 can be described as

   $$\{266(3), 356(6), 366(3), 446(3), 455(3), 456(6), 466(3), 555(1), 556(3), 566(3), 666(1)\}.$$

   The number in parentheses counts the number of permutations of that dice roll, all of which count. For example, 266, 626, and 662 are the permutations of 266. There are 35 possibilities total, so the probability is $\frac{35}{216}$.
3. (*) You flip 3 coins. What is the probability of no heads? one? two? three? Repeat, if you flip four coins.
4. (*) You flip two coins and a die. What is the probability that the number showing on the die exceeds the number of heads you flipped?
5. The following questions concern the dice game of *craps*:

Game Description (Craps): In craps, you roll two dice repeatedly. The rules for the first roll are different than the rules for later rolls.

On the first roll, if you roll a 7 or 11, you win immediately, and if you roll a 2, 3, or 12, you lose immediately. Otherwise, whatever you rolled is called "the point" and the game continues.

If the game continues, then you keep rolling until you either roll 'the point' again, or a seven. If you roll the point, then you win; if you roll a seven (on the second roll or later), you lose.

(a) In a game of craps, compute the probability that you win on your first roll. Conversely, compute the probability that you lose on your second roll. **Solution.**

The probability of winning on your first roll is the probability of rolling a 7 or 11: $\frac{6}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9}$.

For the second question, I intended to ask the probability that you lose on your *first* roll. Oops. Let's answer the question as asked. There are multiple possible interpretations, and here is one. Let us compute the probability that you lose on the second round, presuming that the game goes on to a second round. This is the probability of rolling a 6 or $\frac{1}{6}$.

(b) In a game of craps, compute the probability that the game goes to a second round and you win on the second round. **Solution.** This can happen in one of six possible ways: you roll a 4 twice in a row, a 5 twice in a row, or similarly with a 6, 8, 9, or 10.

The probability of rolling a 4 is $\frac{3}{36}$, so the probability of rolling a 4 twice in a row is $\left(\frac{3}{36}\right)^2$. Similarly with the other dice rolls; the total probability is

$$(1)\quad \left(\frac{3}{36}\right)^2 + \left(\frac{4}{36}\right)^2 + \left(\frac{5}{36}\right)^2 + \left(\frac{5}{36}\right)^2 + \left(\frac{4}{36}\right)^2 + \left(\frac{3}{36}\right)^2 = \frac{9+16+25+25+16+9}{1296} = \frac{100}{1296} = \frac{25}{324}.$$

(c) In a game of craps, compute the probability that the game goes to a second round and you lose on the second round. **Solution.** Multiply the probability that the game goes onto a second round (easily checked to be $\frac{2}{3}$) by the probability $\frac{1}{6}$ computed earlier, so $\frac{1}{9}$.

(d) In a game of craps, compute the probability that you win.

Solution. With probability $\frac{2}{9}$ you win on your first round. We will now compute the probability that you win later, with the point equal to n, for n equal to 4, 5, 6, 8, 9, or 10. We will then add these six results. Write the probability of rolling n on one roll of two dice as $\frac{a}{36}$, so that a is 3, 4, or 5 depending on n.

- As we computed before, the probability of winning on the second round (with point n) is $\left(\frac{a}{36}\right)^2$.
- On each round after the first, there is a probability $\frac{30-a}{36}$ of rolling something other than 7 or the point. This is the probability that the game goes on to another round.
- So, the probability of winning on the third round is the probability of: rolling the point on the first round, going another turn in the second round, rolling the point on the third round. This is $\left(\frac{a}{36}\right)^2 \cdot \left(\frac{30-a}{36}\right)$.
- Similarly, the probability of winning with point n on the fourth round is $\left(\frac{a}{36}\right)^2 \cdot \left(\frac{30-a}{36}\right)^2$, and so on. The total of all these probabilities is

$$\left(\frac{a}{36}\right)^2 \sum_{k=0}^{\infty} \left(\frac{30-a}{36}\right)^k.$$

- For $|r| < 1$, we have the infinite sum formula $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$. Plugging this in, the above expression is

$$\left(\frac{a}{36}\right)^2 \cdot \frac{36}{6+a} = \frac{a^2}{36(6+a)}.$$

So we add this up for $a = 3$ (twice, for $n = 4$ or 5), $a = 4$ (twice), and $a = 5$ (twice). We get

$$2 \cdot \left(\frac{9}{36 \cdot 9} + \frac{16}{36 \cdot 10} + \frac{25}{36 \cdot 11}\right) = \frac{134}{495}.$$

Adding the to the first round probability of $\frac{2}{9}$ we get

$$\frac{2}{9} + \frac{134}{495} = \frac{244}{495}.$$

This is a little less than a half. As expected, the house wins.

6. Consider the game Press Your Luck described above. Assume (despite rather convincing evidence to the contrary) that the show is random, and that you are equally likely to stop on any square on the board.

   (a) On each spin, estimate the probability that you hit a Whammy. Justify your answer.

   (Note: This is mostly not a math question. You have to watch the video clip for awhile to answer it.)

   (b) On each spin, estimate the probability that you *do not* hit a Whammy.

   (c) If you spin three times in a row, what is the probability you don't hit a whammy? Five? Ten? Twenty-eight? (If your answer is a power of a fraction, please also use a calculator or a computer to give a decimal approximation.)

7. Consider the game Rat Race described above.

   (a) Suppose that the contestant only prices one item correctly, and so gets to pick one rat. What is the probability that she wins the car? That she wins *something*? That she wins nothing?

   (b) What if the customer prices all three items correctly? What is the probability that she wins the car? Something? Nothing? All three items?

   (c) Consider now the first part of the game, where the contestant is pricing each item. *Assume* that she has a 50-50 chance of pricing each item correctly. What is the probability she prices no items correctly? Exactly one? Exactly two? All three?

       Comment on whether you think this assumption is realistic.

   (d) Suppose now that she has a 50-50 chance of pricing each item correctly, and she plays the game to the end. What is the probability she wins the car?

8. (*) These questions concern the game of Three Strikes you saw above. A complete analysis is quite intricate but should be possible – an interesting term project possibility.

   (a) How many possibilities are there for the price of the Ferrari?

   (b) How many possibilities are there for the price of the Ferrari, if you know that the first digit is 1 or 2?

   (c) If the contestant draws a number and incorrectly guesses its position, the number is replaced in the bag. What is the probability that the contestant draws five numbers in a row (with no strikes), if she doesn't guess any of their positions correctly?

   (d) If the contestant draws a number and correctly guesses its position, the number is removed from play. What is the probability that the contestant draws five numbers in a row (with no strikes), if she guesses all of their positions correctly?

   (e) (Difficult) Suppose that the contestant somehow knows the price of the car exactly, and never guesses any digit incorrectly. So she must simply draw all the numbers before drawing all of the strikes. What is the probability that she wins the car?

       There is a simple answer to this, but it requires some creativity to find.

   (f) (Very Difficult) Suppose the price of the Ferrari is equally likely to be any of the possibilities starting with 1 or 2. Formulate a good strategy for playing the game, and compute the odds that a contestant using this strategy will win.

   (g) Usually this game is played for more modest cars, that have five-digit prices. How do your answers above change in this case?

   (h) (Video Game History) The game used to be played with different rules: there was only one strike chip instead of three, and it would be replaced in the bag after the contestant drew it. The contestant lost if she drew the strike chip three times.

Re-answer the questions above under these alternate rules. Do these rules make it easier or harder to win?

3 Expectation

We come now to the concept of **expected value**.

We already saw this in the previous clip of *Deal or No Deal*. The contestant chooses one of 26 briefcases, with varying amounts of money. Since the average of the briefcases is around $130,000, we consider this the *expected value* of the game. It is how much a contestant would expect to win on average, if she had the opportunity to play the game repeatedly.

We will give a few simple examples and then give a formal definition.

3.1 Definitions and Examples

Example 1: You play a simple dice game. You roll one die; if it comes up a six, you win 10 dollars; otherwise you win nothing. On average, how much do you expect to win?

Solution. Ten dollars times the probability of winning, i.e.,

$$10 \times \frac{1}{6} = 1.66\ldots$$

So, for example, if you play this game a hundred times, on average you can expect to win 100 dollars.

Example 2: You play a variant of the dice game above. You roll one die; if it comes up a six, you still win 10 dollars. But this time, if it doesn't come up a six, you lose two dollars. On average, how much do you expect to win?

Solution. We take into account both possibilities. We multiply the events that you win 10 dollars or lose 2 dollars and multiply them by their probabilities. The answer is

$$10 \times \frac{1}{6} + (-2) \times \frac{5}{6} = 0.$$

On average you expect to break even.

Here is the formal definition of *expected value*. Please read it carefully, and compare the mathematical formalism to the intuition that you developed by studying the examples above.

Definition 3 (Expected Value): Consider a random process with n possible outcomes. We list:

- The *probabilities* $p_1, p_2, \ldots, p_n$ of each of the events;

- The *values* $a_1, a_2, \cdots, a_n$ of each of the events (in the same order). These are real numbers (positive or negative).

Then, the **expected value** of the random process is

$$\sum_{k=1}^{n} a_k p_k = a_1 p_1 + a_2 p_2 + \cdots + a_k p_k.$$

In our examples, the 'values' will usually represent the amount of money that you win. If it's possible to lose (i.e., in a game of poker), then we indicate this with negative values.

The 'values' don't have to represent monetary amounts. But beware:

Warning Example 4: Today, there is an 80% chance that it will be sunny, and a 20% chance that it will snow. What is the expected value of this process?

Solution. This question can't be answered, unless we assign some sort of numerical value to sunshine and snowing. Do you like snow? How much?

For example, you might say that you rate sunshine a 6 on a scale of 1 to 10, and that you love snow and would rate it a 10. Then the expected value would be

$$0.8 \times 6 + 0.2 \times 10 = 6.8.$$

Even when values do represent money, sometimes expected value computations can be a little bit misleading:

Warning Example 5: You have \$500,000 savings, and you have the opportunity to spend it all to play the following game: You roll four dice. If all of them show a six, you win a billion dollars. Otherwise, you lose your \$500,000. What is the expected value of this game?

Solution. The probability of winning is $\left(\frac{1}{6}\right)^4 = \frac{1}{1296}$, so the probability of losing is $\frac{1295}{1296}$, and your expected value is

$$\frac{1}{1296} \cdot 1000000000 + \frac{1295}{1296} \cdot (-500000) = 271990.74.$$

A positive number, so you expect to win on average. But, for most people, it would be unwise to play this game: even though a billion dollars is $2,000$ times more than \$500,000, it wouldn't have $2,000$ times the effect on your life. There is only so much you can do with lots of money.

Example 6: You roll a die and win a dollar amount equal to your die roll. Compute the expected value of this game.

Solution. The possible outcomes are that you win 1, 2, 3, 4, 5, or 6 dollars, and each happens with probability $\frac{1}{6}$. Therefore the expected value is

$$1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} = \frac{21}{6} = 3.5.$$

When doing expected value computations, you might find it helpful to keep track of all of the necessary information using a table. For example, a table for the above computation might look like the following:

Outcome	Probability	Value
1	$\frac{1}{6}$	1
2	$\frac{1}{6}$	2
3	$\frac{1}{6}$	3
4	$\frac{1}{6}$	4
5	$\frac{1}{6}$	5
6	$\frac{1}{6}$	6

You can then compute the expected value as follows: for each row, multiply the probability and value in each row, and then add the result for all of the rows.

See the end of Section 3.2 for another example of a table in this format.

Example 7: Consider again the Deal or No Deal clip from the introduction:

Link: Deal or No Deal

Skipping ahead to the 16 minute mark, we see that the contestant has been quite lucky. With eleven briefcases left, he has eliminated most of the small briefcases, and most of the largest briefcases remain.

The bank offers him \$141,000 to stop playing. The expected value of continuing is \$237830.54, much higher than the bank's offer. He could consider playing it safe, but if he wants to maximize his expected value then he should continue.

Note, also, that his choice is not actually between \$141,000 and his chosen briefcase. After he eliminates three more briefcases, the banker will make another offer, and so continuing is not *quite* as big of a gamble as it may appear.

The game **Punch a Bunch** from The Price Is Right provides an interesting example of an expected value computation. It will also allow us to introduce the technique of 'backwards induction', which we will return to again later.

Link: The Price Is Right – Punch-a-Bunch

> **Game Description (The Price Is Right – Punch-a-Bunch):** The contestant is shown a punching board which contains 50 slots with the following dollar amounts: 100 (5), 250 (10), 500 (10), 1000 (10), 2500 (8), 5000 (4), 10,000 (2), 25,000 (1). The contestant can earn up to four punches by pricing small items correctly. For each punch, the contestant punches out one hole in the board.
>
> The host proceeds through the holes punched one at a time. The host shows the contestant the amount of money he has won, and he has the option of either taking it and ending the game, or discarding and going on to the next hole.

If you correctly price only one small item, and so get only one punch, there is no strategy: you just take whatever you get. In this case the expected value is the average of all the prizes. This equals the the total value of all the prizes divided by 50, which we compute is $\frac{103000}{50} = 2060$.

In the clip above, the contestant gets three punches. He throws away 500 on his first punch, 1000 on his second, and gets 10,000 on his third. Was he right to throw away the 1000?

We will *assume* that he is playing to *maximize his expected value*. This might not be the case. He might have to pay next month's rent, and prefer the $1,000 prize to a chance at something bigger. He might be going for the glory (he is on national TV, after all), and be willing to risk it all for a chance at a big win. But maximizing expected value is probably a reasonable assumption – and in any case it allows for a mathematical analysis of the show.

If so, then he should clearly throw away the $1,000. In general, the expected value of one punch is $2,060. In the scenario of the clip, it is a little bit higher because two of the cheap prizes are out of play: there is $101,500 in prizes left in 48 holes, for an average of $2,114.58. You don't have to do the math exactly: just remember that two of the small prizes are gone, so the average of the remaining ones goes up slightly.

We will work out an optimal strategy for this game. For simplicity, assume that the contestant gets exactly three punches. As we will see, we want to work *backwards*:

- On your third and last round, there is no strategy: you take whatever you get.
- On your second round, the expected value of your third slot will be close to $2,000. (Unless you drew one of the big prizes in the first or second slot, in which case it will be smaller. But if you draw one of the big prizes, presumably you don't need to be doing a math calculation to figure out that you should keep it.)

  That means that you should throw away the $1,000 prize or anything smaller, and keep a $2,500 prize or anything larger. The $2,500 prize is a fairly close call; the larger prizes should clearly be kept.

- What about your first round? What is the expected value of throwing away your prize, and proceeding to the second round?

To compute this expected value, we refer to our strategy for the second round above. We'll also simplify matters by ignoring the fact that one of the slots is now out of the game and there are only 49 slots remaining. (If we didn't want to ignore this, we would have to do this computation *separately for each different value we might have drawn in the first slot.* This is a pain, and it doesn't actually make much of a difference.)

The following outcomes are possible:

- You might win \$25,000 ($\frac{1}{50}$ chance), \$10,000 ($\frac{2}{50}$ chance), \$5,000 ($\frac{4}{50}$ chance), or \$2,500 ($\frac{8}{50}$ chance). In this case, as we decided earlier you should keep it.
- You might draw one of the other cards, less than \$2,500 ($\frac{35}{50}$ chance). In this case, you should throw it away, and on average you win the expected value of the third punch: approximately \$2,050.

In other words, you should think of the second round as having 15 large prizes, and 35 prizes of \$2,050: imagine that the small prizes are all equal to \$2,060, because that's the expected value of throwing them away and trying again.

Therefore, the expected value of throwing away your prize and going to a second round is

$$25000 \cdot \frac{1}{50} + 10000 \cdot \frac{2}{50} + 5000 \cdot \frac{4}{50} + 2500 \cdot \frac{8}{50} + 2060 \cdot \frac{35}{50} = \$3,142.$$

This means that, on the first round, the contestant should throw away not only the \$1,000 prizes, but also the \$2,500 prizes. So, for example, a contestant playing to maximize her expected value might draw \$2,500 on the first round, throw it away, draw a second \$2,500 prize, and decide to keep that one.

Exercise 8: Consider a Punch-a-Bunch contestant who wins four punches. What is her optimal strategy on every round?

Explain why her strategy for the second, third, and fourth rounds is the same as described above, and determine her ideal strategy on the first round.

Exercise 9: The rules for Punch-a-Bunch have changed somewhat over the years. For example, here is a clip from 2003:

Link: The Price Is Right – \$10,500 on Punch-a-Bunch

In this playing, the top prize was \$10,000, but several of the cards have 'Second Chance' written on them: the contestant may immediately punch another slot and add that to the 'Second Chance' card.

How would rule changes affect the analysis above?

As we mentioned earlier, our analysis illustrates the process of 'backwards induction'. In a game with several rounds, analyze the last round first and then work backwards.

This is especially relevant in *competitive* games. Many game shows feature a single player at a time, playing against the studio. But sometimes multiple players are directly competing against each other for money or prizes. A chess player will certainly consider how her opponent will respond before making a move. It is the same on game shows.

We now consider some expected value computations arising from the popular game show **Wheel of Fortune**.

Game Description (Wheel of Fortune): Three contestants play several rounds where they try to solve word puzzles and win money. Whoever wins the most money also has the opportunity to play in a bonus round.

The puzzle consists of a phrase whose letters are all hidden. In turn, each contestant may either **attempt to solve the puzzle** or **spin the wheel**. If the contestant attempts to solve: a correct guess wins the money that he has banked; an incorrect guess passes play to the next player.

The **wheel** has wedges with varying dollar amounts or the word 'bankrupt'. If the contestant spins and lands on 'bankrupt', he loses his bank and his turn. Otherwise, he guesses a letter: if it appears in the puzzle, it is revealed and the contestant wins the amount of his spin for each time it appears. Otherwise, play passes to the next contestant.

There are also other rules: the contestants can 'buy a vowel'; the wheel has a 'lose a turn' space which doesn't bankrupt the contestant; and so forth.

Link: Wheel of Fortune

In the clip above, Robert wins the first round in short order. After guessing only two letters (and buying a vowel) he chooses to solve the puzzle. Was his decision wise?

Remark: The notes below are from my presentation in Fall 2016. In Spring 2018, I didn't review these notes closely, and instead I redid the computation on the fly at the board.

Note that the results are different! This reflects, among other things, the difficulty of modeling this game show accurately. You might find it useful to compare and contrast what's written here with what was presented in class.

Let us make some assumptions to simplify the problem and set up an expected value computation:

- Robert wants to maximize the expected value of his winnings this round.

This is not completely accurate, especially in the final round; the contestants are interested in winning *more than the other two contestants*, because the biggest winner gets to play the bonus round. But it is reasonably close to accurate, especially early in the running.

- Robert definitely knows the solution to the puzzle.

  So, if he chooses to spin again, it's to rack up the amount of prizes and money he wins.

- If Robert loses his turn, then he won't get another chance and will therefore lose everything.

  In fact, there is a chance that each of the other two contestants will guess wrongly or hit the 'bankrupt' or 'lose a turn' spots on the wheel. But this puzzle doesn't look hard: the first word *don't* is fairly obvious; also, the second word looks like *bet*, *get*, or *let* and B, G, and L are all in the puzzle. Robert is wise to assume he won't get another chance.

- We won't worry too much about the 'weird' spots on the board.

  The $\frac{1}{3}$-sized million dollar wedge is not what it looks like: it sits over (what I believe is) a \$500 wedge now, and offers the contestant the opportunity to win \$1,000,000 in the bonus round *if* he goes to the bonus round *and* doesn't hit bankrupt before then *and* solves the bonus puzzle correctly *and* chooses the million dollars randomly as one of five prizes. It's a long shot, although three contestants have indeed won the million.

So we freeze-frame the show and we count what we see. Out of 24 wedges on the wheel, there are:

- 16 ordinary money wedges on the wheel, with dollar amounts totalling \$12,200.
- Two 'bankrupt' wedges, a 'lose a turn' wedge, and an additional two thirds of a bankrupt wedge surrounding the million.
- A one-third size wedge reading 'one million'.
- The cruise wedge. This isn't relevant to the contestant's decision, because he wins the cruise and reveals an ordinary wedge underneath. We can't see what it is, so let's say \$500.
- Two other positive wedges.

Let us now compute the expected value of another spin at the wheel. There are (with the cruise wedge) 17 ordinary wedges worth a total of \$12,700. If the contestant hits 'bankrupt' or 'lose a turn' he loses his winnings so far (\$10,959 including the cruise). Let us guess that the million wedge is worth, on average, \$5,000 to the contestant and that the other two are worth \$2,000 each. His expected value from another spin is

$$\frac{1}{24} \cdot 12700 + \frac{3\frac{2}{3}}{24} \cdot (-10959) + \frac{2}{24} \cdot 2000 + \frac{\frac{1}{3}}{24} \cdot 5000 = -\$909.01.$$

Under the above assumptions, it is clear by a large margin to solve the puzzle and lock in his winnings.

Remark: You may be wondering where the $\frac{1}{24} \cdot 12700$ came from. Here is one way to see it: the seventeen wedges have an average of $\frac{12700}{17}$ dollars each, and there is a $\frac{17}{24}$ probability of hitting one of them. So the contribution is

$$\frac{12700}{17} \times \frac{17}{24} = \frac{12700}{24}.$$

Now let us suppose that there was some consonant appearing in the puzzle twice. In that case Robert would know that he could guess it and get *double* the amount of money he spun. So, in our above computation, we double the 12700. (We should probably increae the 2000 and 5000 a little bit, but not double them. For simplicity's sake we'll leave them alone.) In this case the expected value of spinning again is

$$\frac{1}{24} \cdot 12700 \cdot 2 + \frac{3\frac{2}{3}}{24} \cdot (-10959) + \frac{2}{24} \cdot 2000 + \frac{\frac{1}{3}}{24} \cdot 5000 = -\$379.84.$$

It still looks like Robert shouldn't spin again, although if some constant appeared three times, then it might be worth considering.

For an example where Robert arguably chooses unwisely, skip ahead to 10:45 on the video (the third puzzle) where he solves the puzzle with only \$1,050 in the bank. In the exercises, you are asked to compute the expected value of another spin. Note that there are now two L's and two R's, so he can earn double the dollar value of whatever he lands on. There is now a \$10,000 square on the wheel, and hitting 'bankrupt' only risks his \$1,050. (His winnings from the first round are safe.)

There is one factor in favor of solving now: an extra prize (a trip to Bermuda) for the winner of the round. If it were me, I would definitely risk it. You do the math, and decide if you agree.

(But see the fourth run, where I would guess he knows the puzzle and is running up the score.)

Who Wants To Be a Millionaire?

Here is a typical clip from Who Wants To Be a Millionaire:

Link: Who Wants To Be a Millionaire?

The rules in force for this episode were as follows.

Game Description (Who Wants to be a Millionaire?): The contestant is provided with a sequence of 15 trivia questions, each of which is multiple choice with four possible answers. They are worth an increasing amount of money: 100, 200, 300, 500, and then (in thousands) 1, 2, 4, 6, 16, 32, 64, 125, 250, 500, 1000. (In fact, in this epsiode, the million dollar question was worth $2,060,000.)

At each stage he is asked a trivia question for the next higher dollar amount. If he answers correctly, he advances to the next level; he can also decline to answer, in which case his winnings are equal to the value of the last question he answered.

In general, an incorrect answer forfeits the contestant's winnings. At the $1,000 and $32,000 level his winnings are protected: if he reaches the $1,000 stage, he will win $1,000 even if he later answers a question incorrectly, and the same is true of the $32,000 level.

He has three 'lifelines', each of which may be used exactly once over the course of the game: '50-50', which eliminates two of the possible answers; 'phone a friend', allowing him to call a friend for help; and 'ask the audience', allowing him to poll the audience for their opinion.

In general, the early trivia questions are easy but the later ones are quite difficult.

Here is the general question we want to ask.

Question 10 (WWTBAM – General Version): A contestant on the show has correctly answered the question worth x dollars, and is trying to decide whether or not to guess an answer to the next question.

Suppose that he estimates his probability of answering correctly at δ. Should he guess or not?

We have formulated the question quite generally, with two variables x and δ. We could, for example, have asked: 'A contestant is pondering answering the $64,000 question, and he estimates his odds of answering correctly at 50%. Should he guess or not?' By introducing variables we have generalized the question. It turns out that we will want to study individual values of x one at a time, but we can work with all δ simultaneously. So, instead of an answer of yes or no, an answer might take the following shape: 'For the $64,000 question, the contestant should guess if $\delta > 0.4$.'

In other words, whether the contestant should guess depends on how confident he is. Sounds reasonable enough!

We'll introduce some simplifying assumptions. For starters, let's skip to the interesting part of the show and assume that $x \geq 32000$. Indeed, we may then assume that $x \geq 64000$: if the contestant has answered the $32,000 question correctly, then he has 'locked in' the $32,000 prize and risks nothing by continuing.

For starters, we will consider the following additional simplification which *only considers the next question*.

Question 11 (WWTBAM – $125,000, next question only): The contestant has answered the $64,000 question correctly, and he is considering whether to answer the $125,000 question.

Assume that, if he answers it correctly, he will walk away with the $125,000 prize. Should he guess?

Note that there is a possibility we are ignoring. Maybe, if he correctly guesses the $125,000 question, the contestant will be asked a $250,000 question that he definitely knows the answer to. And so, in that (unlikely?) case, he will be able to win $250,000 without taking on any additional risk.

We should also think about the form we expect our answer to take. Since the question involves the variable δ, the answer probably should too. In particular, we don't expect an answer of the form 'Yes, he should guess' or 'No, he should walk away'. Rather, we might expect an answer roughly along the following lines: 'The contestant should guess if he has at least a 50-50 shot, and otherwise he should walk away.' Or, to say the same thing more mathematically, 'The contestant should guess if and only if $\delta \geq \frac{1}{2}$.'

Solution. We must determine whether (i.e., for what values of δ) the expected value of guessing is greater than the contestant's current $64,000.

Since the contestant will receive $32,000 for a wrong answer and $125,000 for a correct one, and these events will occur with probability $1-\delta$ and δ respectively, the expected value of guessing is

$$(1-\delta) \cdot 32000 + \delta \cdot 125000 = 32000 + \delta \cdot 93000.$$

When is this greater than 64000? Let's do the algebra:

$$\begin{aligned}(1-\delta) \cdot 32000 + \delta \cdot 125000 &> 64000 \\ 32000 + 93000\delta &> 64000 \\ 93000\delta &> 32000 \\ \delta &> \frac{32000}{93000} = \frac{32}{93}.\end{aligned}$$

So the contestant should guess if and only if he believes that his probability of guessing correctly is at least $\frac{32}{93}$, approximately 34%. In particular, random guessing would be bad, but (for example) if the contestant can eliminate two of the answers, it makes sense for him to guess.

Question 12 (WWTBAM – $250,000, next question only): The contestant has now answered the $125,000 question correctly, and is considering whether to answer the $250,000 question.

Assume that, if he answers it correctly, he will walk away with the $250,000 prize. Should he guess?

This is the same question with different numbers, and we can solve it similarly. We should expect the cutoff for δ to go up a little bit: he is still playing to (approximately) double his money, but now he is risking (approximately) three quarters of his winnings rather than half.

The inequality to be solved is now

$$32000 + \delta \cdot (250000 - 32000) > 125000,$$

and doing the algebra yields $\delta > \frac{93}{218}$, which is about 42%.

Now, we consider the possibility that the contestant might be able to answer two questions at a time:

> **Question 13 (WWTBAM – $250,000, next two questions):** Assume again that the contestant answered the $125,000 question correctly, and is considering whether to answer the $250,000 question. Again he has a probability δ of guessing correctly.
>
> This time, assume that there is a 40% chance that the $500,000 question will be one he knows. If he guesses the $250,000 question correctly, and gets a $500,000 question that he doesn't know, he will walk away with $250,000. If he guesses the $250,000 question correctly, and gets a $500,000 question that he does know, he will answer it and then walk away with $500,000.
>
> Should he guess?

Solution. We first compute the expected value of reaching the $250,000 level. Since there is a 40% chance he will move up further to the $500,000 level, this expected value is

$$0.4 \times 500000 + 0.6 \times 250000 = 350000.$$

Therefore, the expected value of guessing on the current question is

$$(1 - \delta) \cdot 32000 + \delta \cdot 350000 = 32000 + \delta \cdot 318000.$$

Comparing this to the $125,000 that he keeps if he walks away, we see that he should guess if and only if

$$32000 + 318000\delta > 125000,$$

which (try the algebra yourself!) is satisfied when $\delta > \frac{93}{318}$, or about 29%. He should still not randomly guess (probability $\delta = \frac{1}{4}$, but if his guess is even slightly better than random, then he would maximize his expected value by guessing.

Continuing this line of reasoning further, the contestant could get very lucky and get two questions in a row that he definitely knows. We leave the analysis of this last possibility as an exercise:

Exercise 14: Assume yet again that the contestant answered the \$125,000 question correctly, and is considering whether to answer the \$250,000 question. Again he has a probability δ of guessing correctly.

This playing also offers \$500,000 and \$1,000,000 questions. For each, the contestant anticipates a 40% probability of a question he definitely knows the answer to. If he gets such a question, he will answer it; otherwise, he will walk away with his current winnings.

For what values of δ should the contestant be willing to guess? In particular, if the contestant has no idea and must guess randomly with $\delta = \frac{1}{4}$, should he do so?

3.2 Linearity of expectation

Example 15: You roll two dice and win a dollar amount equal to the sum of your die rolls. Compute the expected value of this game.

Solution. (**Hard Solution**). The possible outcomes and the probabilities of each are listed in the table below.

2	3	4	5	6	7	8	9	10	11	12
$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

The expected value is therefore

$$2\times\frac{1}{36}+3\times\frac{2}{36}+4\times\frac{3}{36}+5\times\frac{4}{36}+6\times\frac{5}{36}+7\times\frac{6}{36}+8\times\frac{5}{36}+9\times\frac{4}{36}+10\times\frac{3}{36}+11\times\frac{2}{36}+12\times\frac{1}{36}$$
$$= \frac{2+6+12+20+30+42+40+36+30+22+12}{36} = \frac{252}{36} = 7,$$

or exactly 7 dollars.

You should always be suspicious when you do a messy computation and get a simple result.

Solution. (**Easy Solution**). If you roll one die and get the dollar amount showing, we already computed that the expected value of this game is 3.5.

The game discussed now is equivalent to playing this game twice. So the expected value is $3.5 \times 2 = 7$.

Similarly, the expected value of throwing a thousand dice and winning a dollar amount equal to the number of pips showing is (exactly) \$3,500.

Here are a couple examples where the expected 'value' refers to something other than money.

Example 16: You flip ten coins. On average, how many heads do you expect to flip?

Solution. Consider each coin on its own. On average, you expect to flip half a head on it: with probability $\frac{1}{2}$ it will land heads, and otherwise it won't.

So, the expected number of heads is ten times this, or $10 \times \frac{1}{2} = 5$.

If you are more comfortable thinking of expected values in terms of money, imagine that the contestant wins a dollar for each heads, and the computation is the same.

As a similar example, you roll a thousand dice. On average, how many sixes do you expect to roll? The answer is $1000 \times \frac{1}{6} = \frac{500}{3}$, or about 167.

Here is a more sophisticated problem that illustrates the same principle.

Example 17: Consider once again the game of Rat Race. Suppose that our contestant gets to pick two out of five rats, that first place wins a car (worth \$16,000), that second place wins meal service (worth \$2,000) and that third place wins a guitar (worth \$500).

What is the expected value of the game?

The hard solution would be to compute the probability of every possible outcome: the contestant wins the car and the meals, the car and the guitar, the guitar and the meals, the car only, the meals only, the guitar only, and nothing. What a mess!!! Instead, we'll give an easier solution.

Solution. Consider only the first of the contestant's rats. Since this rat will win each of the three prizes for the contestant with probability $\frac{1}{5}$, the expected value of this rat's winnings is

$$16000 \times \frac{1}{5} + 2000 \times \frac{1}{5} + 500 \times \frac{1}{5} = 3700.$$

The second rat is subject to the same rules, so the expected value of its winnings is also \$3700. Therefore, the total expected value is $\$3,700 + \$3,700 = \$7,400$.

Indeed, the expected value of the game is \$3,700 per rat won, so this computation gives the answer no matter how many rats she wins.

There is a subtlety going on in this example, which is noteworthy because we **didn't** worry about it. Suppose, for example, that the first rat fails to even move from the starting line. It is a colossal zonk for the contestant, who must pin all of her hopes on her one remaining rat. Does this mean that her expected value plummets to \$3,700? *No!* It now has a one in *four* chance of winning each of the three remaining prizes, so its expected value is now

$$16000 \times \frac{1}{4} + 2000 \times \frac{1}{4} + 500 \times \frac{1}{4} = 4625.$$

Conversely, suppose that this rat races out from the starting block like Usain Bolt, and wins the car! Then the expected value of the remaining rat goes *down*. (It has to: the car is off the table, and the most it can win is \$2,000.) Its expected value is a measly

$$2000 \times \frac{1}{4} + 500 \times \frac{1}{4} = 625.$$

This looks terribly complicated, because **the outcomes of the two rats** are not independent. If the first rat does poorly, the second rat is more likely to do well, and vice versa.

The principle of **linearity of expectation** says that our previous computation **is correct, even though the outcomes are not independent.** If the first rat wins the car, the second rat's expected value goes down; if the first rat loses or wins a small prize, the second rat's expected value goes up; and these possibilities average out.

Theorem 18 (Linearity of Expectation):
Suppose that we have a random process which can be broken up into two or more separate processes. Then, the total expected value is equal to the sum of the expected values of the smaller processes.

This is true whether or not the smaller processes are independent of each other.

Often, games can be broken up in multiple ways. In the exercises you will redo the Rat Race computation a different way: you will consider the expected value of winning just the car, just the meals, and just the guitar – and you will verify that you again get the same answer.

We can now compute the expected value of Rat Race as a whole! Recall that Rat Race begins with the contestant attempting to price three small items correctly, and winning one rat for each item that she gets right.

Example 19: Assume for each small item, the contestant has a 50-50 chance of pricing it correctly. Compute the expected value of playing Rat Race.

Solution. Recall from your homework exercises that the probability of winning zero, one, two, or three rats is respectively $\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{8}$, and $\frac{1}{8}$. Since the expected value of Rat Race is \$3.700 per rat won, the expected value of the race is respectively \$0, \$3,700, \$7,400, and \$11,100. Therefore the expected value of Rat Race is

$$0 \times \frac{1}{8} + 3700 \times \frac{3}{8} + 7400 \times \frac{3}{8} + 11000 \times \frac{3}{8} = 5550.$$

Although this solution is perfectly correct, it misses a shortcut. We can use linearity of expectation twice!

Solution. Each attempt to win a small item has probability $\frac{1}{2}$ of winning a rat, which contributes \$3,700 to the expected value. Therefore the expected value of each attempt is $3700 \times \frac{1}{2} = 1850$. By linearity of expectation, the expected value of three attempts is

$$1850 + 1850 + 1850 = 5550.$$

As a reminder, if you like, you might choose to keep track of everything using a table. For example, you could describe the first solution to the above example as follows:

Outcome	Probability	Value
0 rats	$\frac{1}{8}$	0
1 rats	$\frac{3}{8}$	3700
2 rats	$\frac{3}{8}$	7400
3 rats	$\frac{1}{8}$	11000

As before, you compute the expected value by multiplying the probability and value in each row, and adding all the rows.

3.3 Some classical examples

Finally, we treat several classical examples of expected value computations.

The St. Petersburg Paradox. You play a game as follows. You start with \$2, and you play the following game. You flip a coin. If it comes up tails, then you win the \$2. If it comes up heads, then your stake is doubled and you get to flip again. You keep flipping the coin, and doubling the stake for every flip of heads, until eventually you flip tails and the game ends.

How much should you be willing to pay to play this game?

To say the same thing another way, your winnings depend on the number of consecutive heads you flip. If none, you win \$2; if one, you win \$4; if two, you win \$8, and so on. More generally, if you flip k consecutive heads before flipping tails, you win 2^{k+1} dollars. Unlike most game shows, you never risk anything and so you will certainly continue flipping until you flip tails.

We first compute the probability of every possible outcome:

- With probability $\frac{1}{2}$, you flip tails on the first flip and win \$2.
- With probability $\frac{1}{4}$, you flip heads on the first flip and tails on the second flip: the probability for each is $\frac{1}{2}$ and you multiply them. If this happens, you win \$4.
- With probability $\frac{1}{8}$, you flip heads on the first two flips and tails on the third flip: the probability for each is $\frac{1}{2}$ so the probability is $\left(\frac{1}{2}\right)^3$. If this happens, you win \$8.

- Now, we'll handle all the remaining cases at once. Let k be the number of consecutive heads you flip before flipping a tail. Then, the probability of this outcome is $\left(\frac{1}{2}\right)^{k+1}$: we've made $k+1$ flips and specified the result for each of them.

  Your winnings will be 2^{k+1} dollars: you start with \$2, and you double your winnings for each of the heads you flipped.

We now compute the expected value of this game. This time there are infinitely many possible outcomes, but we do the computation in the same way. We multiply the probabilities by the expected winnings above, and add:

$$\$2 \cdot \frac{1}{2} + \$4 \cdot \frac{1}{4} + \$8 \cdot \frac{1}{8} + \$16 \cdot \frac{1}{16} + \cdots = \$1 + \$1 + \$1 + \$1 + \cdots = \infty$$

The expected value of the game is infinite, and you should be willing to pay an infinite amount of money to play it. This does not seem to make sense.

By contrast, consider the following version of the game. It has the same rules, only the game has a maximum of 100 flips. If you flip 100 heads, then you don't get to keep playing, and you're forced to settle for 2^{101} dollars, that is, \$2,535,301,200,456,458,802,993,406,410,752.

The expected value of *this* game is a mere

$$\$2\cdot\frac{1}{2}+\$4\cdot\frac{1}{4}+\$8\cdot\frac{1}{8}+\$16\cdot\frac{1}{16}+\cdots+\$2^{100}\cdot\frac{1}{2^{100}}+\$2^{101}\cdot\frac{1}{2^{100}} = \$1+\$1+\$1+\$1+\cdots+\$1+\$2 = \$102.$$

Now think about it. If you won the maximum prize, and it was offered to you in \$100 bills, it would weigh[2] 2.5×10^{25} kilograms, in comparison to the weight of the earth which is only 6×10^{24} kilograms. If you stacked them, you could reach any object which has been observed anywhere in the universe.

This is ridiculous. The point is that *the real-life meaning of expected values can be distorted by extremely large, and extremely improbable, events.*

The Coupon Collector Problem. This well-known problem is often referred to as the 'coupon collector problem':

> **Example 20:** To get people into their store, Amy's Burrito Shack begins offering a free toy with every burrito they serve. Suppose that there are four different toys, and with every order you get one of the toys at random.
>
> You want to collect all four toys. On average, how many burritos do you expect that you'll have to eat?

The answer is more than four: once you have at least one toy, the store could disappoint you by giving you a toy you already have. You'll just have to come back tomorrow, eat another burrito, and try again.

[2] more precisely: have a mass of

McDonald's has long given out toys with their Happy Meals, and this question might also be relevant to anyone who has collected baseball, Magic, or Pokemon cards.

McDonald's has also run games along these lines, based on the *Scrabble* and *Monopoly* board games. For example, in *Monopoly*, with each order you got a game piece representing a 'property'; the properties are color-coded, and if you collect all properties of one color, then you win a prize. However, McDonald's made sure that, for each color, one of the properties was much rarer than the others – which changes the math considerably.

The game *Spelling Bee* on The Price Is Right also has a similar mechanic – for example, see the following video:

Link: The Price Is Right – Spelling Bee

But there the 'C', the 'A', and the 'R' don't have the same frequency, and also there are 'CAR' tiles – making the analysis more complicated.

We start with a warmup question, the solution of which will be part of our solution to the Coupon Collector Problem.

Example 21: A jar has a number of marbles, some of which are blue. You draw marbles from the jar until you draw a blue one. If each time there is a probability of δ of drawing a blue marble, on average how many marbles will you draw?

Note that we assume this probability δ never changes; for example, this would be true if we replaced each marble after we drew it.

This will be an expected value computation, and it illustrates another computation where the 'value' doesn't refer to money, but in this case to the number of draws required. We will offer two different strategies to solve this problem. Although the first one can eventually be made to work, it is sort of a mess, and so after understanding the principle we will move on to the second.

Solution. The game could take any number of turns. Write $P(n)$ for the probability that the game ends after exactly n draws.

- We have $P(1) = \delta$, the probability of drawing blue on the first draw.
- We have $P(2) = (1 - \delta)\delta$: you have to *not* draw blue on the first draw, and then draw blue on the second draw.
- We have $P(3) = (1 - \delta)^2\delta$: this time you have to not draw blue on the first two draws, and then draw blue.
- Similarly, we have $P(4) = (1 - \delta)^3\delta$, and so on: we have $P(n) = (1 - \delta)^{n-1}\delta$.

So our expected value is given by the infinite sum

$$E = 1 \cdot \delta + 2 \cdot (1 - \delta)\delta + 3 \cdot (1 - \delta)^2\delta + 4 \cdot (1 - \delta)^3\delta + \cdots$$

If you know the right tricks, you might be able to compute the infinite sum. (You can also *approximate it*, for individual values of δ, by using a calculator to compute the first few terms.) But we will abandon this solution in favor of a more interesting one.

Solution. *Watch carefully: this will feel like cheating! It's not!*

Write E for the answer, the expected number of draws required. Consider the first marble. If it's blue, then the game is over and you win. If it's not blue, then you wasted your move. Since the situation is exactly the same as when you started, on average it will take you E more moves to finish.

This is enough to write down a formula for E. We have

$$E = \delta \cdot 1 + (1 - \delta) \cdot (1 + E).$$

This follows from what we described earlier: with probability δ, you're done after the first move, and with probability $1 - \delta$, you're not done and on average need E more moves.

This might look useless: it's a formula for E in which E also appears on the right. But we can solve it for E:

$$\begin{aligned} E &= \delta \cdot 1 + (1 - \delta) \cdot (1 + E) \\ E &= \delta + 1 + E - \delta - \delta E \\ 0 &= 1 - \delta E \\ \delta E &= 1, \end{aligned}$$

and so our solution is

$$E = \frac{1}{\delta}.$$

This illustrates an important principle: describing an unknown quantity E in terms of itself looks like it would lead us in circles, but in this case it was a useful thing to do.

Solution (Coupon Collector Problem): We work one toy at a time, this time going *forwards* instead of backwards.

First toy. On your first visit to the store, you are guaranteed to get a new toy since you don't have any toys yet. The expected number of visits is 1.

Second toy. This is the same as the marbles problem, so we can incorporate the solution we already gave. Each time you visit the store, you have a probability $\frac{3}{4}$ of getting a toy that is different than what you have. So, on average, the number of visits required is $\frac{1}{3/4} = \frac{4}{3}$.

Third toy. This time, you have a probability of $\frac{1}{2}$ of getting a new toy on each visit to the store, so the average number of visits required is $\frac{1}{1/2} = 2$.

Fourth toy. By the same reasoning, the average number of visits required is $\frac{1}{1/4} = 4$.

Using linearity of expectation, we can add the average number of visits required to obtain each new toy:

$$1 + \frac{4}{3} + 2 + 4 = 8\frac{1}{3}.$$

If there are n different toys to collect, then you can work out that the average number of visits required is

$$n\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right).$$

If you know calculus, you might recognize the sum inside the parentheses: the *natural logarithm* function $\ln(x)$ satisfies

$$\ln(x) = \int_1^x \frac{1}{t}\, dt,$$

and the sum inside the parentheses is a reasonably good approximation to the area under the graph of $\frac{1}{t}$ between $t = 1$ and $t = n$. So, the average number of visits required is approximately

$$n \ln(n).$$

Remark: Note that the above is only an *approximation*, and *not* an exact formula. There is no nice exact formula for the sum $1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

There is, however, a better approximation, namely

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \approx \ln(n) + \gamma,$$

where the **Euler-Mascheroni constant** γ is

$$\gamma = 0.55721\ldots,$$

and more precisely is defined by

$$\gamma = \lim_{n\to\infty}\left(-\ln(n) + \sum_{k=1}^{n} \frac{1}{k}\right).$$

Understanding precisely why all this works is an interesting (and rather challenging) exercise in calculus.

An Umbrella Problem.

Example 22: (**The Umbrella Problem – Expected Value Version**)

One hundred guests attend a party. It is raining, and they all bring umbrellas to the party. All of their umbrellas are different from each other.

At the end of the party, the host hands umbrellas back to the guests at random. On average, how many guests will get their own umbrella back?

Solution. This is another expected value problem, where here the 'value' will be 'the number of guests getting their own umbrella'.

We solve this using linearity of expectation. *Consider only one guest at a time.* The probability that she gets her own umbrella is $\frac{1}{100}$. Therefore, the average number of guests that get their own umbrella, *considering only that one guest*, is $\frac{1}{100}$.

We now use linearity of expectation – we can add this expected value of $\frac{1}{100}$ over all the guests, even though these probabilities are not independent. The total expected number of guests that get their own umbrella is therefore

$$100 \cdot \frac{1}{100} = 1.$$

It may seem strange to consider an 'average number of guests', thinking about only one guest at a time. But this is like the example where we considered the expected number of coin flips that came up heads. Here, this number is 0 if she doesn't get her umbrella back, and 1 if it does. The 'average number of guests' equals the probability that she gets her own umbrella.

Later, we will consider a variation of the umbrella problem: given the same situation, what is the probability that *no one* gets their own umbrella back?

This is not an expected value computation, so we can't give a similar solution. It turns out that we need an advanced counting principle known as *inclusion-exclusion*, which we'll come to later. The solution (surprisingly!) will also use a bit of calculus.

3.4 Exercises

1. (Warmup exercise, no need to hand in, feel free to skip.) You toss ten dice. Compute the expected value of:

   (a) The number of sixes;

   (b) The number of dice which land on two or five;

   (c) The number of pips (dots) showing;

   (d) Your payoff, if you lose \$10 for each 1, but otherwise get the amount of your die roll;

(e) (A bit more tricky) Your payoff, if you lose \$10 for each 1, otherwise get the amount of your die roll, and get an additional \$50 if none of the ten dice shows a 1;

(f) (A bit more tricky) Times you toss two sixes in a row (assuming you toss the dice one at a time).

We now proceed to the 'real' exercises, to be handed in:

2. Watch the Deal or No Deal clip from the introduction. Fast forward through all the talk and choosing briefcases if you like, but pay attention to each time the bank offers him a buyout to quit. Compute, in each case, the expected value of playing the game out until the end. Does the bank ever offer a payout larger than the expected value?

   What would you decide at each stage? Explain.

3. Consider again a game of Rat Race with two rats, played for prizes worth \$16,000 (car), \$2,000 (meals), and \$500 (guitar).

   (a) Compute the expected value of the game, considering only the car and ignoring the other prizes. (This should be easy: she has a 2 in 5 chance of winning the car.)

   Solution. She has a $\frac{2}{5}$ chance of winning the car, so the answer is $\frac{2}{5} \times 16000 = 6400$.

   (b) Compute the expected value of the game, considering only the meals.

   Solution. As above, the answer is $\frac{2}{5} \times 2000 = 800$.

   (c) Compute the expected value of the game, considering only the guitar.

   Solution. As above, the answer is $\frac{2}{5} \times 500 = 200$.

   (d) By linearity of expectation, the expected value of the game is equal to the sum of the three expected values you just computed. Verify that this sum is equal to \$7,400, as we computed before.

   Solution. $6400 + 800 + 200 = 7400$.

4. Do the exercise posed at the end of the discussion of *Who Wants to be a Millionaire.*

The next questions concern the Price is Right game **Let 'em Roll**. Here is a clip:

Link: The Price Is Right – Let 'em Roll

Game Description (Let 'em Roll – The Price Is Right):

The contestant is given one or more chances to roll five dice. Each die has \$500 on one side, \$1,000 on another, \$1,500 on a third, and a car symbol on the other three. If a car symbol shows on each of the five dice, she wins the car. Otherwise, she wins the total amount of money showing on the dice without car symbols.

The contestant may get up to three rolls, depending on whether she correctly prices two small grocery items. After each roll other than her last, she may choose to: (1) set the dice with a car symbol aside, and reroll only the rest; or (2) quit, and keep the money showing on the dice without car symbols.

5. First, consider a game of Let 'em Roll where the contestant only gets one dice roll.

   (a) Compute the probability that she wins the car.

   (b) Compute the expected value of the game, considering the car and ignoring the money. (The announcer says that the car is worth \$16,570.)

   (c) Compute the expected value of the game, considering the money and ignoring the car.

   (d) Compute the total expected value of the game.

   Solution. The probability that she wins the car is $(\frac{1}{2})^5 = \frac{1}{32}$: there are five dice, and each must show a car.

   Considering only the car, the expected value of the game is $\frac{1}{32} \times 16570 \sim \$518$.

   Considering only the money, each die contributes an expected value of

$$\frac{1}{6} \times 500 + \frac{1}{6} \times 1000 + \frac{1}{6} \times 1500 = 500.$$

   Since there are five dice, the total is \$2500, and the total (including both car and dice) is \$3018.

6. (a) Now watch the contestant's playing of the game, where after the second round she chooses to give up \$2,500 and reroll. Compute the expected value of doing so. Do you agree with her decision?

   (b) Suppose that after two turns she had rolled no car symbols, and \$1,500 was showing on each of the five dice. Compute the expected value of rerolling, and explain why she should *not* reroll.

   (c) Construct a hypothetical situation where the expected value of rerolling is within \$500 of not rerolling, so that the decision to reroll is nearly a tossup.

Solution. After her second round, she has three cars (which she would keep if she rerolls) and \$2,500. If she rerolls, she has a one in four probability of winning the car, so her expected value from the car is $\frac{1}{4} \times 16570 \sim 4142$. She also obtains an additional expected value of \$1000 from the money, for a total of \$5142. As this is much larger than \$2,500, rerolling is a good idea if she can stomach some risk.

In the second scenario, the expected value is the same as the one-turn version (because she will reroll everything): \$3,018. Since this is much less than \$7,500, it is a good idea to keep the money.

Here is an intermediate scenario. Suppose two cars are showing and she rerolls the other three dice. Then the expected value of the game is

$$\frac{1}{8} \times 16570 + 3 \times 500 \sim 3571.$$

So if the three money dice are showing a total of \$3,500, it is essentially a tossup decision whether or not to reroll.

As another correct solution, suppose only one car is showing and she rerolls the other four.

$$\frac{1}{16} \times 16570 + 4 \times 500 \sim 3035.$$

If the four money dice are showing \$3000 total, once again it is approximately a tossup.

Yet another correct solution has no cars showing and low amounts of money on the dice: a total of either \$2500 or \$3000.

7. If the contestant prices the small grocery items correctly and plays optimally, compute the expected value of a game of Let 'em Roll.

   (Warning: if your solution is simple, then it's wrong.)

   Solution. (See the appendix.)

3.5 Appendix: The Expected Value of Let 'em Roll

This was Problem #7 on the homework, which was good for extra credit (even for a partial solution.) It's definitely challenging. You have to work backwards, sort of like Punch-a-Bunch, only with more expected value computations. We will also introduce a few approximations to make the computations simpler.

Step 1. After two dice. Suppose that n dice are showing monetary amounts, and you reroll them. Then the expected value of doing so is

$$\left(\frac{1}{2}\right)^n \cdot 16570 + 500n,$$

which is given in the following table (rounded to the nearest dollar).

Dice	EV
0	16570
1	8785
2	5143
3	3571
4	3035
5	3017

You should reroll if and only if your expected value is larger than the amount of money showing. This can happen when $n = 3$, $n = 4$, or $n = 5$.

Step 2. After one die.

- If you have five cars, you win (EV 16570).
- If you have four cars, you should clearly reroll, the expected value is

$$\frac{1}{2} \cdot 16570 + \frac{1}{2} \cdot 8785 = 12678.$$

- If you have three cars, again you should clearly reroll. The expected value is

$$\frac{1}{4} \cdot 16570 + \frac{1}{2} \cdot 8785 + \frac{1}{4} \cdot 5143 = 9821.$$

- If you have two cars, and you reroll, there is now a small chance you will walk away after your second roll. This will occur only if you roll no cars and more than 3571 dollars. The probability of this happening is $\frac{4}{216}$; the four outcomes we count are (15, 15, 15), (15, 15, 10), (15, 10, 15), and (10, 15, 15). The expected value of rerolling is

$$\frac{1}{8} \cdot 16570 + \frac{3}{8} \cdot 8785 + \frac{3}{8} \cdot 5143 + \left(\frac{1}{216} \cdot 4500 + \frac{3}{216} \cdot 4000 \right) + \left(\frac{1}{8} - \frac{4}{216} \right) \cdot 3571 = 7751.$$

  You should still always reroll.

- We skip to the case where you roll no cars at all. Since this is unlikely (probability $\frac{1}{32}$), it doesn't make much of a contribution to the expected value, and we can afford a course approximation.

  Your chances of winning the car are pretty low, and simultaneously there is a lot of cash showing on the board. The expected value of each die is now \$1000 (since we're assuming it's not showing a car, and hence is showing 500, 1000, or 1500), so the total expected value on the board is \$5000. Taking it looks pretty good. Let us say that you almost always take it, unless there's a lot less than \$5000 showing, in which case you do a little bit better by going for the car. We'll round the expected value of this case up to \$5500.

- Finally, suppose you roll one car. The expected value of rerolling is

$$\frac{1}{16} \cdot 16570 + \frac{4}{16} \cdot 8785 + \frac{6}{16} \cdot 5143 + \frac{4}{16} \cdot x + \frac{1}{16} \cdot y,$$

  where x and y are the expected values if you get one and no cars, respectively.

  If you get one car, the expected value of rerolling is 3571, and we'll reroll unless the dice show a total of 4000 or 4500. This is a similar computation to what we did before. For simplicity, let's just round 3571 up to 3700, since it's unlikely the total will be 4000 or 4500.

  If you get no cars, the expected value of rerolling is 3035, and the expected value of the dice is 4000. So you'll take whatever money is showing on the dice, unless there's less than 3000. So we'll start with 4000, and round up to 4200 to account for the fact that we'll reroll all the outcomes below 3035.

  We have

$$\frac{1}{16} \cdot 16570 + \frac{4}{16} \cdot 8785 + \frac{6}{16} \cdot 5143 + \frac{4}{16} \cdot 3700 + \frac{1}{16} \cdot 4200 = 6348.$$

So, finally, we can use the above to compute the expected value of the entire game:

$$\frac{1}{32} \cdot 16570 + \frac{5}{32} \cdot 12678 + \frac{10}{32} \cdot 9821 + \frac{10}{32} \cdot 7751 + \frac{5}{32} \cdot 6348 + \frac{1}{32} \cdot 5500 = 9154.$$

4 Counting

For an event E in a sample space S, in which all outcomes are equally likely, the probability of E was defined by the formula

$$P(E) = \frac{N(E)}{N(S)}.$$

In order to use this formula, we need to compute $N(E)$ and $N(S)$ – the number of outcomes enumerated by the event, and the number of outcomes in the sample space as a whole. One way to do this is to simply list them. But what if E and S are very large?

We discussed the addition and multiplication rules for probability, as convenient shortcuts available in some cases. Nevertheless, sometimes we will want to count $N(E)$ and $N(S)$. Indeed, we saw an example of this in Section 2.3, where we learned to count permutations. The sample space consisted of all the *permutations* of a fixed *string*, and we saw that *if T is a string with n symbols, all distinct, then there are $n!$ permutations of T.*

Here we will further develop our toolbox of counting techniques, with applications to computing probabilities.

4.1 The Addition and Multiplication Rules

As counting is closely related to probability, we should not be surprised to see analogues of the addition and multiplication rules.

Theorem 1 (The Addition Rule for Counting): Suppose that A and B are *disjoint* sets. Then,

$$N(A \cup B) = N(A) + N(B).$$

Here $A \cup B$ is the *union* of A and B: the set of elements which are in either.

Theorem 2 (The Multiplication Rule for Counting): Suppose that S is a set whose elements can be described in two steps, such that:

- There are r possibilities for the first step;
- There are s possibilities for the second step, no matter how the first step is carried out.

Then,

$$N(S) = rs.$$

The rule also works for sets requiring more than two steps: as long as the number of possibilities for each step doesn't depend on the previous steps, you just multiply together the number of possibilities for each step.

It probably isn't clear what this means yet! Here is a basic example.

Example 3: How many strings are there consisting of one letter, followed by one digit?

Solution. This is an application of the multiplication rule, where the 'steps' outline how you write down such a string.

First, choose a letter; this can be done in 26 ways. Then, choose a digit, this can be done in 10 ways. So the total number of strings is $26 \cdot 10 = 260$.

Here is an example illustrating more than two steps.

Example 4: In South Carolina, a license tag can consist of any three letters followed by any three numbers. (A typical example might be TPQ-909. Ignore the dash in the middle.)

How many different license tags are possible?

Solution. Here there are six steps: there are 26 possibilities for the first letter, 26 for the second, and 26 for the third; similarly there are 10 possibilities for each of the three digits. So the total number of possibilities is $26^3 \cdot 10^3 = 17576000$.

You don't *have* to divide it into six steps; for example, you could choose the string of three letters first (there are 17,576 possibilities), and then the string of three digits (there are 1,000 possibilities), and $17576 \cdot 1000 = 17576000$. The *answer* above (that is, the number 17576000) is uniquely determined, but the solution isn't.

Another thing you can do, if you like, is vary the order of the steps. You can choose the digits first, and then the letters, and $10^3 \cdot 26^3$ is also 17576000. It doesn't really matter.

Here is an example where we can't use the multiplication rule.

Warning Example 5: How many strings of three digits are there, where each digit is larger than the previous one?

Solution. We *try* to use the muliplication rule. In the first step, choose the first digit; there are 10 possibilities. But how many possibilities are there for the second digit? It depends on what first digit you chose! So we can't proceed in this way.

You can also verify that it doesn't help to choose the last digit first, or to choose the middle digit first. No matter what you do, the number of possibilities for the remaining digits depend on what you chose for the first digit.

There is a nice method for solving problems like this, but we won't discuss it yet.

Here is a somewhat similar example, but where you *can* use the multiplication rule.

Example 6: How many license tags are possible which don't repeat any letters or digits?

Solution. There are still 26 possibilities for the first letter, and now 25 for the second and 24 for the third. At each step, we must avoid the letters that were previously used. Similarly there are 10, 9, and 8 possibilities for the three digits. The total number of possibilities is

$$26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 = 11232000.$$

Notice, that when we go to pick the second letter, the *set* of possibilities depends on the first choice made: it consists of the entire alphabet, with the first letter removed. But the *number* of possibilities will always be 25, and so we can use the multiplication rule.

These computations may be used to solve probability questions. For example:

Example 7: What is the probability that a random license tag doesn't repeat any letters or numbers?

This follows from the previous two computations. The result is

$$\frac{11232000}{17576000} = .639\ldots$$

Here are a few examples involving permutations.

Example 8: On a game of Ten Chances, Drew Carey feels particularly sadistic and puts all ten digits – zero through nine – to choose from in the price of the car. The price of the car consists of five different digits. How many possibilities are there?

Solution. There are 10 possibilities for the first digit, 9 for the second, 8 for the third, 7 for the fourth, and 6 for the fifth, for a total of

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 30240$$

possibilities.

Good luck to the poor sucker playing this game. A car is not likely to be in their future.

Example 9: In the above example, suppose you know that the first digit is not zero. Now how many possibilities are there?

Solution. This time there are only 9 possibilities for the first digit (anything but zero). There are also 9 possibilities for the second (anything but whatever the first digit was). For the remaining digits there are similarly 8, 7, 6 choices for the last three in turn. The total is

$$9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 27216.$$

Example 10: In the previous example, suppose you know that the first digit is not zero **and** that the last digit is zero. Now how many possibilities are there?

We will look at two possible solutions. The first solution will lead us into a dead-end, and we will need to abandon it and try a different method.

Solution attempt. We begin as in the previous example. There are 9 possibilties for the first digit, 9 for the second, 8 for the third, and 7 for the fourth.

How many possibilities for the last digit are there? *It depends!* It is 1 if we haven't used the zero already, but if we have then the number of possibilities is 0. Since the answer can never be 'it depends' in the multiplication rule, we abandon this solution and try again.

Solution. We can answer this question correctly by choosing the digits in a *different order*. First, choose the first digit first – it can be anything other than the zero, 9 ways), then the last digit (must be the zero, so 1 way), and then the second, third, and fourth digits in turn (8, 7, and 6 ways), for a total of

$$9 \cdot 8 \cdot 7 \cdot 6 = 3024$$

ways.

Alternatively, we could have picked the last digit before the first, and we can pick the second, third, and fourth digits in any order. It is usually best to find one order which works and stick to it.

One final example along these lines:

Example 11: In the previous example, suppose you know that the first digit is not zero **and** that the last digit is either zero or five. Now how many possibilities are there?

Solution. This requires a little bit more work. Since the first and last digits depend on each other, we consider all the ways to choose them. If the last digit is 0, then the first digit can be anything else, so there are nine such possibilities. If the last digit is 5, then the first digit can be anything other than 0 or 5, so there are eight such possibilities. By the addition rule, the first and last digit can be chosen together in 17 different ways.

Now, we solve the rest of this problem as before. The second, third, and fourth digits can be chosen in 8, 7, and 6 ways respectively, no matter how the first and last were chosen. So the total number of possibilities is

$$17 \cdot 8 \cdot 7 \cdot 6 = 5712.$$

4.2 Permutations and combinations

We first recall the definition of a **permutation**, and also introduce the variant of an r-**permutation**.

Definition 12 (Permutations and r-permutations): A **permutation** of a string T is any reordering of T.

An r-**permutation** of a string T is any reordering of r of the symbols from T.

So, for example consider the string $T = 12334$. Some permutations of T are 32314, 24133, and 12334. Some 3-permutations of T are 314, 332, and 412.

Remark: Suppose that T is a string with n symbols. Then r-permutations of T exist only when $0 \leq r \leq n$. There is exactly one 0-permutation of T, which is the empty string.

Note also that, in a string with n symbols, n-permutations of T are the same thing as permutations.

Definition 13 ($P(n,r)$): We write $P(n,r)$ for the number of r-permutations of a string with n distinct symbols.

The number $P(n,r)$ doesn't depend on the particular string, as long as it has n distinct symbols.

Indeed, we can compute a formula for $P(n,r)$.we already saw that $P(n,n) = n!$; this is our claim that a string with n distinct elements in it has $n!$ permutations. In more generality, we have the following:

Theorem 14 (Counting r-permutations): We have

$$P(n,r) = \frac{n!}{(n-r)!}.$$

To see this, use the multiplication rule! When we construct an r-permutation, there are n possibilities for the first symbol, $n-1$ possibilities for the second, and so on: one less for each subsequent symbol. There are $n-r+1$ possibilities for the rth symbol: we start at n and count down by 1 $r-1$ times. So we have

$$P(n,r) = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdots (n-r+1).$$

This can also be written as

$$P(n,r) = \frac{n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdots (n-r+1) \cdot (n-r) \cdot (n-r-1) \cdots 2 \cdot 1}{(n-r) \cdot (n-r-1) \cdots 2 \cdot 1} = \frac{n!}{(n-r)!}$$

The following table lists $P(n,r)$ for all $0 \leq r \leq n \leq 6$. (Each row corresponds to a fixed value of n, and each row corresponds to a fixed value of r.)

	0	1	2	3	4	5	6
$n=0$	1	-	-	-	-	-	-
$n=1$	1	1	-	-	-	-	-
$n=2$	1	2	2	-	-	-	-
$n=3$	1	3	6	6	-	-	-
$n=4$	1	4	12	24	24	-	-
$n=5$	1	5	20	60	120	120	-
$n=6$	1	6	30	120	360	720	720

The dashes indicate that $P(n,r)$ is not defined when $r > n$. (Alternatively, we could sensibly define these values of $P(n,r)$ to be zero.)

It is a worthwhile exercise to stop reading, and to recreate this table for yourself from scratch. Even more worthwhile is to look for patterns, and then figure out how to explain them. Some patterns you can observe:

- $P(n,r)$ increases when you increase either n or r. If you increase n, then you have more symbols to choose from; if you increase r, then you make more choices (in addition to your previous ones).
- $P(n,1) = n$. This counts the number of ways to choose 1 symbol, when you have n to choose from. So of course it is n.
- $P(n,n) = P(n,n-1)$. If you want to turn an $(n-1)$-permutation of a length n string into an n-permutation, then there is exactly one way to do so: there's one symbol left, and you stick it on the end.
- **Discover and describe your own!**

Note that all these patterns are a consequence of the *formula* for $P(n,r)$, but when possible it is a good idea to think about the *definition* of $P(n,r)$, like we did above.

Combinations. Combinations are like permutations, only the order doesn't matter. We will give two definitions, and you should convince yourself that they are the same.

Definition 15 (Combinations (1)): Let T be a string with n distinct symbols. Then an **r-combination of** T is any choice of r symbols from T. Unlike permutations, the order doesn't matter: two r-combinations are considered the same if they have the same elements, even if the order is different.

Definition 16 (Combinations (2)): Let T be a set with n elements. Then an **r-combination of** T is any subset of T consisting of r of these elements.

The first definition emphasizes the relationship to permutations, and the second definition reflects how we will usually think about combinations. The only difference is whether we think of our 'symbols' or 'elements' as being arranged in a string, or simply placed in a set.

Notice that we only consider strings with *distinct* elements. We could drop this requirement in the *first* definition, but it wouldn't make sense in the second. In set theory, it is conventional that *sets are defined exclusively in terms of what elements belong to them.* So order and repetitions don't matter. For example, each of the following is the same set as $\{1, 2, 3\}$:

$$\{3, 1, 2\}, \{3, 2, 1\}, \{2, 3, 2, 1, 3\}, \{1, 1, 1, 3, 1, 2, 3\}.$$

We will introduce the following notation for counting these:

Definition 17 ($C(n, r)$): We write $C(n, r)$ or $\binom{n}{r}$ for the number of r-combinations of an n-element set.

The latter notation is read "n choose r", and is ubiquitous in mathematics. These numbers are also called 'binomial coefficients', for reasons that we'll explain later.

Example 18: Write out all the 3-combinations of 12345.

Solution. 123, 124, 125, 134, 135, 145, 234, 235, 245, and 345. There are ten.

We could have written 321 or $\{1, 2, 3\}$ (for example) in place of 123, and we would have described the same combination.

Example 19: Write out all the 2-combinations of 12345.

Solution. 45, 35, 34, 25, 24, 23, 15, 14, 13, and 12. Again, there are ten.

There are the same number, and indeed this is no coincidence.

To see what's going on, we can arrange them in a table:

123	45
124	35
125	34
134	25
135	24
145	23
234	15
235	14
245	13
345	12

We've listed all the 3-combination in the left column, and the 2-combination in the right column. They way we've matched them up, you can see a one-to-one correspondence (a **bijection**): the second combination lists all the elements that were left out of the first combination, and vice versa. In each row, each of the elements 1, 2, 3, 4, 5 appears exactly once.

Indeed, if you think about this more, there will also be a similar correspondence between r-combinations and $(n-r)$-combinations. So, the following is true.

Theorem 20 (r- and $(n-r)$-combinations): For any n and r we have

$$C(n,r) = C(n,n-r).$$

Soon we will see the general formula for $C(n,r)$, and we could also explain the above theorem using the formula. But explaining why it is true directly in terms of the definition is more interesting!

To try and predict the formula for $C(n,r)$, we will write out all the 2-permutations and all the 2-combinations of the string 12345. The 2-combinations, as before, are

$$45, 35, 34, 25, 24, 23, 15, 14, 13, 12.$$

In fact, much earlier we also listed the 2-permutations of this string, although we didn't call them that. In our discussion of *Rat Race*, we listed all the possibilities for the positions of the pink and orange rats, in that order. They were the 2-permutations of 12345:

$$12, 21, 13, 31, 14, 41, 15, 51, 23, 32, 24, 42, 25, 52, 35, 53, 45, 54$$

There are 20 of them – twice as many as 2-permutations. This is because every 2-combinations appears twice in the list of 2-combinations – once in the same order, and once reversed.

Using the same sort of reasoning, we can explain the following general formula for $C(n,r)$.

Theorem 21 (Formula for $C(n,r)$): We have

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Since this theorem is very important, we want to understand why it's true. Since $P(n,r) = \frac{n!}{(n-r)!}$, the theorem is equivalent to explaining why

$$C(n,r) = P(n,r)/r!, \tag{2}$$

or after rearranging,

$$P(n,r) = C(n,r) \cdot r!. \tag{3}$$

We will pretend that we know how to count $C(n,r)$ already, and use this information to explain how to count $P(n,r)$. This is (3), and this is what we saw in our 2-combinations example just above. Since we already have a formula for $P(n,r)$, we get a formula for $C(n,r)$.

Finally, explaining (3) is the easy part! The right side counts all the ways to produce an r-permutation, in two steps:

- First, we choose r of our n symbols to appear in the r-permutation, without worrying about the order we choose them in. By definition, there are $C(n,r)$ ways to do this.
- Having chosen r symbols, we choose an order for them, which is the same as a permutation of them. There are $r!$ ways to do this.

So (3) follows by the multiplication rule!

Here is an example of how we can use this.

Example 22: You flip four coins. What is the probability that exactly two of them are heads?

Solution. One way is to list all the possible outcomes of the coin flips. Any sequence of heads and tails is equally likely. The possibilities are:

$$HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT,$$
$$THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT.$$

There are sixteen possibilities total. (We knew this by the multiplication rule: $16 = 2 \times 2 \times 2 \times 2$.) Of these, 6 have two heads, so the answer is

$$\frac{6}{16} = \frac{3}{8}.$$

Solution. The solution above is good enough for four coins, but what if we don't want to list all the possibilities? For example, what if we flip eight coins? There will be $2^8 = 256$ possibilities; it would take a long time to list them all.

Let us re-list all the possibilities, but in a different way. We will describe each outcome by saying *which coin flips came up heads*. For example, we will describe $HHTH$ as $\{1, 2, 4\}$. If we do so, the list of possibilities looks like this:

$$\{1,2,3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,2\}, \{1,3,4\}, \{1,3\}, \{1,4\}, \{1\},$$
$$\{2,3,4,\}, \{2,3\}, \{2,4\}, \{2\}, \{3,4\}, \{3\}, \{4\}, \{\}.$$

We can count that there are 6 possibilities with exactly two heads – the number of subsets of $\{1, 2, 3, 4\}$ with exactly two elements.

But wait! This is $C(4, 2)$. We could have just used our formula! We have

$$C(4,2) = \frac{4!}{2!2!} = \frac{24}{2 \times 2} = 6.$$

This last solution works very well even if we flip more coins:

Example 23: You flip ten coins. What is the probability that exactly five land heads?

Solution. There are $2^{10} = 1024$ total possible outcomes. Of these, $C(10, 5)$ of them have five heads, and

$$C(10,5) = \frac{10!}{5! \cdot 5!} = 252.$$

So the probability of obtaining five heads is

$$\frac{252}{1024} \approx 24.6\%.$$

Remark: We have

$$C(10,5) = \frac{10!}{5! \cdot 5!} = \frac{3628800}{120 \times 120}.$$

The arithmetic can get unwieldy, at least without a calculator.

A simpler way to do this is to note that $\frac{10!}{5!} = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$. So, we have

$$C(10,5) = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}.$$

Now, look for things that cancel. For example, we have

$$C(10,5) = \frac{10 \cdot 9 \cdot \not{8} \cdot 7 \cdot 6}{5 \cdot \not{4} \cdot 3 \cdot \not{2} \cdot 1} = \frac{\overset{2}{\not{10}} \cdot 9 \cdot \not{8} \cdot 7 \cdot \overset{2}{\not{6}}}{\not{5} \cdot \not{4} \cdot \not{3} \cdot \not{2} \cdot 1} = 2 \cdot 9 \cdot 7 \cdot 2 = 252.$$

Still a bit messy, but not too terrible.

4.3 Plinko and Pascal's Triangle

This section has two aims, which we will achieve simultaneously:

- To introduce the Price Is Right game *Plinko*, perhaps the most popular game to appear on the show – and to analyze its strategy.
- To compute a table for $C(n,r)$, just as we did before for $P(n,r)$.

  Since we have $C(n,r) = C(n, n-r)$, the table will have a nice symmetry, and it will be natural to write it down in a triangle. This table is known as *Pascal's Triangle*, and we will explore some of its many interesting properties.

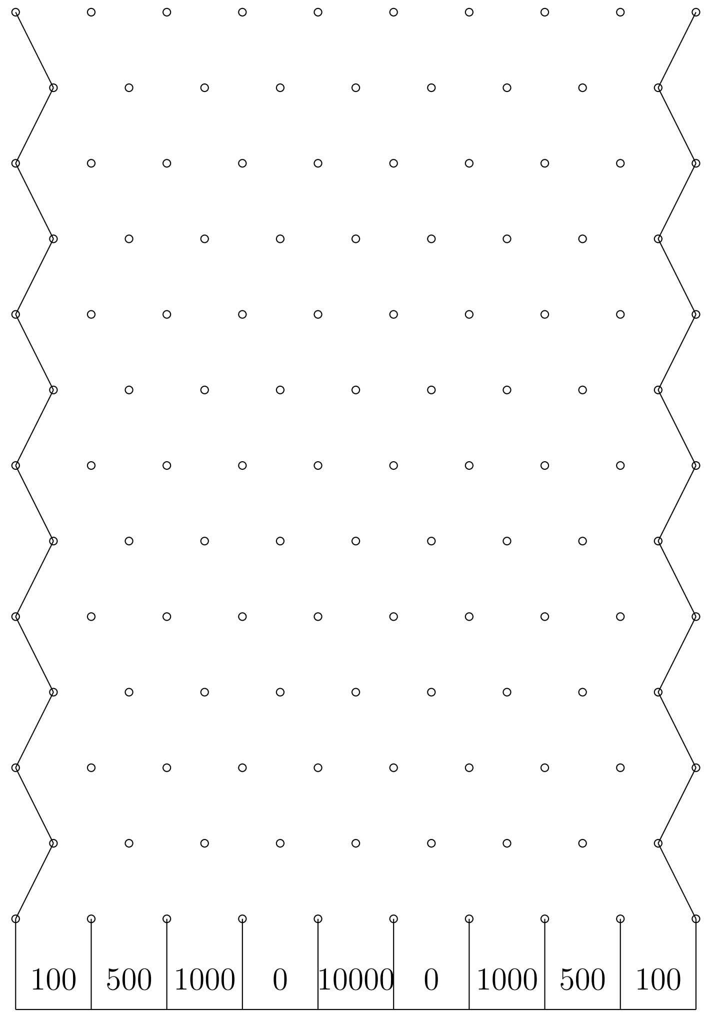
Here is a typical playing of **Plinko**:

Link: The Price Is Right – Plinko

Game Description (Plinko – The Price Is Right): The contestant has the opportunity to win up to five *Plinko chips* and then drop each of them down a board. The board has a number of pegs in the middle, and eventually each chip will land in one of the slots in the bottom. The slots are marked with monetary amounts up to $10,000, with $10,000 in the middle, and the contestant wins prizes corresponding to whatever slots her chips land in.

The question is, **where should the contestant drop her pucks**?

Here is a graphical representation of a Plinko board.

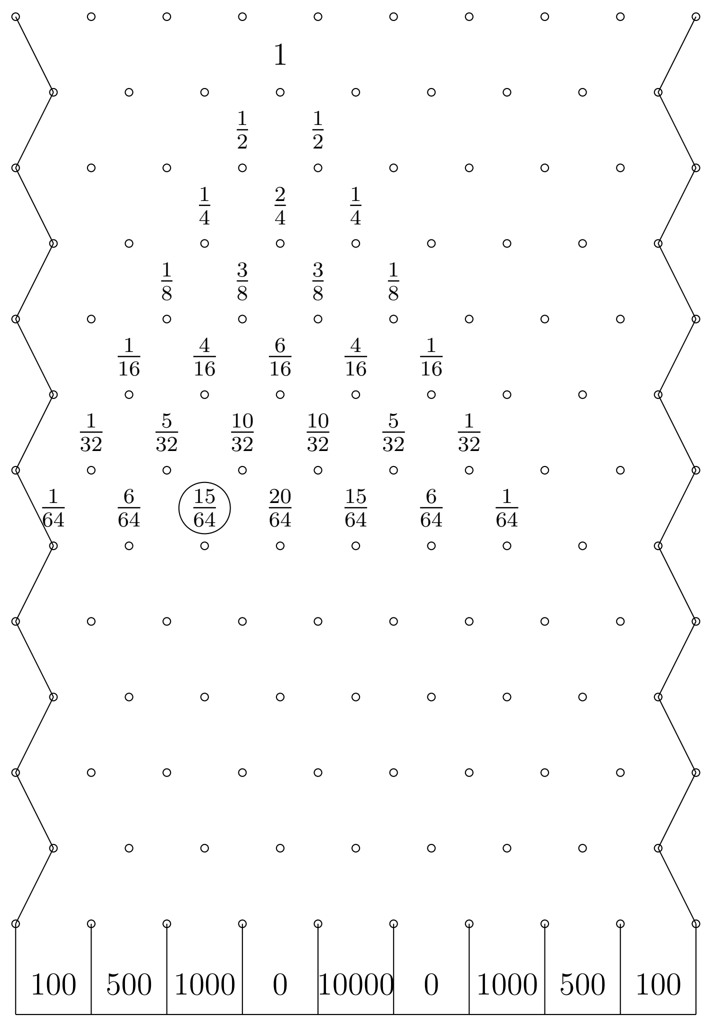


There are nine slots open at the top of the board, and each puck will eventually land in one of the nine spots at the bottom.

We have to model the behavior of the puck. We will assume that **when the puck hits a peg, it goes to its immediate left or immediate right, with probability $\frac{1}{2}$ each.** As an obvious exception, if it hits a wall, it gets forced back to the center of the board.

If you watch enough clips from the show, you will see that this assumption isn't entirely true: sometimes it behaves erratically and skips over pegs. If we ignore this possibility, we can build a simpler mathematical model which is nearly correct.

Subject to this assumption, we can now compute the probability that it lands in any given slot! We assume that the contestant has dropped it one slot to the left from the center.



Here, **each fraction represents the probability that the puck goes through that spot on the grid.** We have started to compute all the probabilities, and we will keep going.

But before we do: how do we compute these? For example, near the top, we labeled the fourth peg with a 1 – the puck is certain to go into whatever slot you drop it into. We labeled the two pegs below that with $\frac{1}{2}$ – with a 50-50 chance, the puck will either go to the left or to the right.

How about the numbers below? You have to compute each row one at a time, because the probabilities for each row depends on the row above it. As an example, let's compute the circled probability in the bottom row, third from the left.

A puck passing through this spot could have come from the peg above and to the *left*, labeled with $\frac{5}{32}$, and it could have come from the peg above and to the *right*, labeled with $\frac{10}{32}$:

- The probability that the puck passes through the left peg above is $\frac{5}{32}$, and if it does, then the probability that it goes to the circled spot is $\frac{1}{2}$. So the probability that both of these events happen is

$$\frac{5}{32} \times \frac{1}{2} = \frac{5}{64}.$$

- Similarly, the probability that the puck passes through the right peg above is $\frac{10}{32}$, and so the probability that it does so and then travels left, to the $\frac{15}{64}$ spot, is

$$\frac{10}{32} \times \frac{1}{2} = \frac{10}{64}.$$

Since a puck passing through the circled spot could have come from either of the two pegs above it, the probability that it reaches these spots is the sum of the two probabilities above, or

$$\frac{5}{64} + \frac{10}{64} = \frac{15}{64}.$$

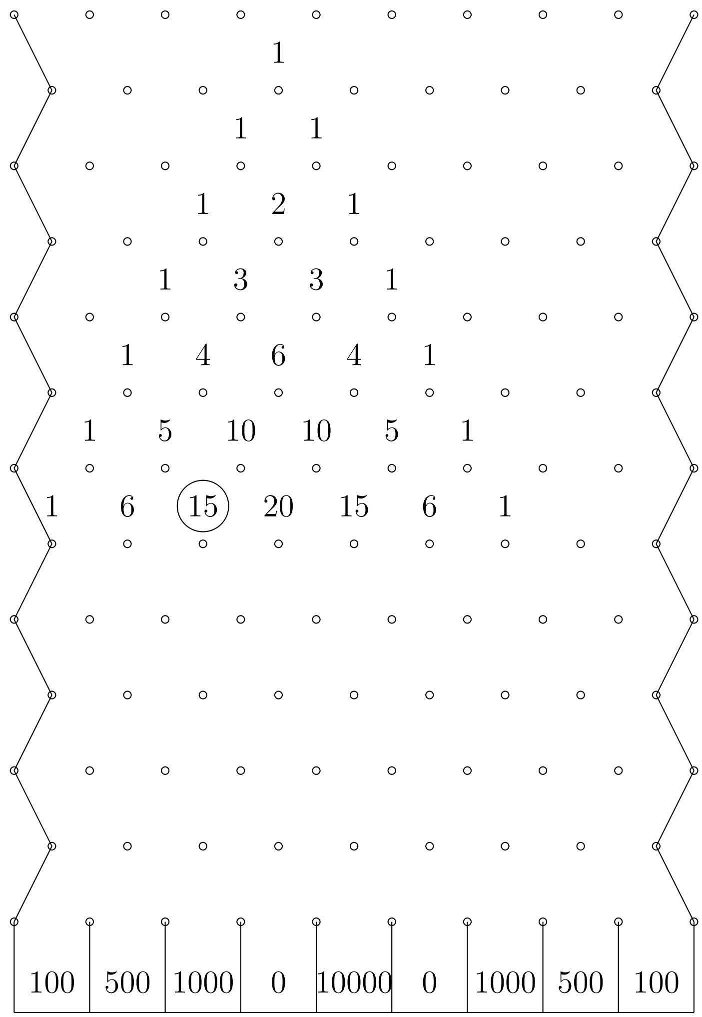
Indeed, we see that **to compute each probability, you add the two probabilities above it, and then divide by 2.** The blank spots are all zeroes – these are locations that the puck can't possibly reach.

When we keep going, things change a bit – because of the wall. For example, the probability on the left, beneath the $\frac{1}{64}$ and the $\frac{6}{64}$, is

$$\frac{1}{64} + \frac{1}{2} \times \frac{6}{64} = \frac{8}{128}.$$

We multiply $\frac{6}{64}$ by $\frac{1}{2}$, because a puck reaching this spot can go either right or left. But a puck reaching the $\frac{1}{64}$ spot *must* go right.

But before we continue, we will change our perspective a little bit by *keeping only the numerators of the fractions*. We can see above, for each n, that the nth row consists of fractions with 2^{n-1} in the denominator, and so this is equivalent to multiplying the nth row by 2^{n-1}. If we do so, then the above diagram looks like this:

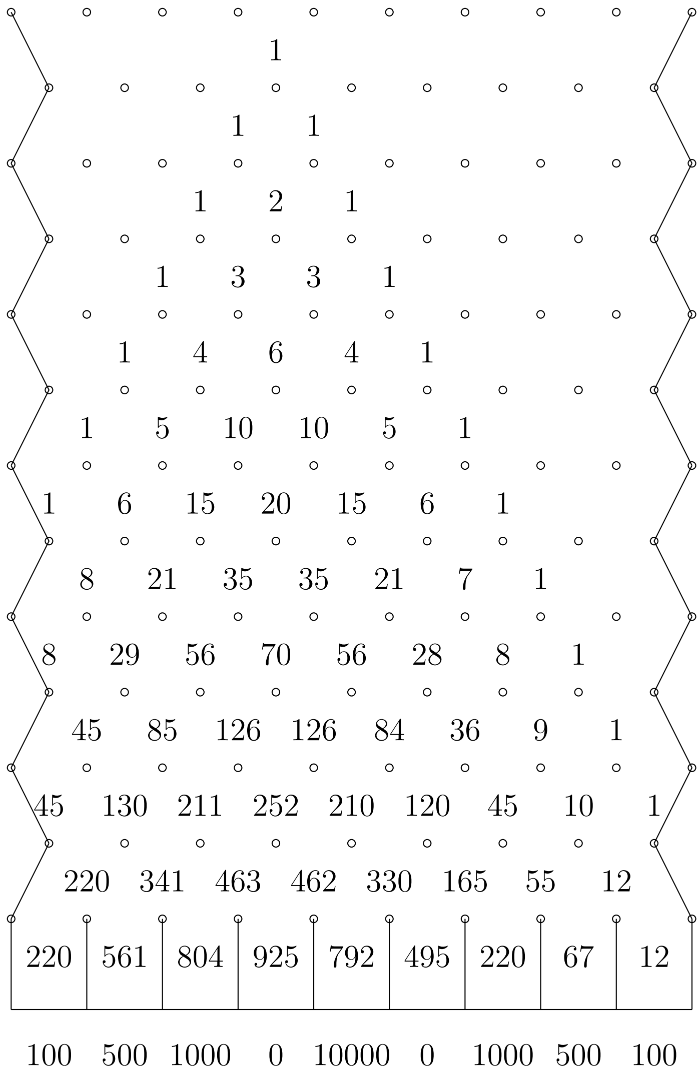


Before, to compute each probability, we added the two probabilities above it, and divided by 2. Since we are removing the denominators, we no longer divide by 2: *each number is simply the sum of the two numbers above it.*

These numbers have a meaning of their own: instead of the *probability* of reaching a given spot, these numbers represent the *number of ways* to reach a given spot. For example, how many ways are there to reach the circled spot (the same spot we circled last time)? There are 5 different ways a puck could have gotten to the spot above and to the left, and 10 different ways the puck could have gotten to the spot above and to the right, and so 15 different ways the puck could have gotten here.

Indeed, we can list them as sequences of L's and R's (standing for *left* and *right* respectively). The fifteen possible combinations of L's and R's are:

$$LLLLRR, LLLRLR, LLRLLR, LRLLLR, RLLLLR,$$
$$LLLRRL, LLRLRL, LRLLRL, RLLLRL,$$
$$LLRRLL, LRLRLL, RLLRLL, LRRLLL, RLRLLL, RRLLLLL.$$



Finally, we continue the pattern we saw before. *The numbers at the side of the board, from which you can't go both left and right, we have to double before adding to the next row.* This means that they no longer represent the number of ways to get to a particular spot. But they do still represent probabilities, if we divide by successive powers of 2.

The numbers in the bottom row add to $2^{12} = 4096$, and to compute the corresponding probabilities you divide by 4096. (The probabilities have to add to 1, because you know the puck has to land *somewhere.*) **The most likely row is the row directly below where you dropped the puck.** You have a $\frac{792}{4096}$ probability of landing your puck in the \$10,000 slot, which is a little more than 19%, and you can as an exercise figure the expected value of this puck drop.

Our diagram is nearly symmetric – indeed it *was* symmetric until we ran into the walls. To see this symmetry further, let's consider a simplified, hypothetical version of the game, where *the walls of the game don't exist.* We assume that the gameboard keeps going off infinitely far to the left and the right.

Indeed, if a puck hits one of the walls, it's unlikely to make it all the way back to the center (and the big money), so in a way this assumption is not so off the mark.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 330 165 55 11 1

1 12 66 220 495 792 924 792 495 220 66 12 1

220 561 804 925 792 495 220 67 12

100 500 1000 0 10000 0 1000 500 100

For comparison's sake we have included the previous numbers (where the board had walls) below the final numbers. They are different, but not *so* different.

We have just written out the first thirteen rows of **Pascal's Triangle**.

Definition 24 (Pascal's Triangle): Pascal's Triangle is the triangular table explained above, which starts with the thirteen rows above and continues forever.

You can produce it by the following steps:

- The top row has a solitary 1 in it.

- Each row has one more number than the previous, with a 1 at each edge. **Each number in the middle of the table is equal to the sum of the two above it.**
- By convention the rows are numbered as follows: the top row is the **zeroth** row. After that, the rows are numbered 1, 2, 3, etc., and the nth row starts with a 1 and an n.

4.4 Properties of Pascal's Triangle

Pascal's Triangle is really quite a miraculous object. We now want to explain some of its most striking properties.

Theorem 25 (Pascal's Triangle – Sum of nth row): The numbers in the nth row of Pascal's Triangle sum to 2^n.

We observed this before, as we were constructing it: each number contributes twice to the row below it: once to its left, and once to its right. Hence, each row must sum to double the previous row.

Remember that we counted the top row as the zeroth row – it sums to $2^0 = 1$.

Theorem 26 (Pascal's Triangle Entries): The numbers of the nth row of Pascal's Triangle are exactly $C(n,0)$, $C(n,1)$, ..., $C(n,n)$ in order.

So, in other words, Pascal's Triangle is our long-promised table of $C(n,r)$.

We explained this in our coin flip example above. Alternatively, think back to our count of the fifteen sequences of L's and R's with two R's and four L's. Our list started

$$LLLLLL, LLLRLR, LLRLLR, LRLLLR, \ldots$$

We can associate a 2-combination of $\{1,2,3,4,5,6\}$ to each: Just list what two positions have the R's. Alternatively, we can associate a 4-combination to each: list what two positions have the L's.

We also saw that each number in Pascal's Triangle is the sum of the two above it. When we remember that Pascal's Triangle is a table of $C(n,r)$, we get the following:

Theorem 27 (Sum Formula for Combinations): We have

$$C(n,r) = C(n-1,r-1) + C(n-1,r)$$

whenever n and r are positive integers.

As another way of illustrating this theorem, we offer the following (somewhat strange) example.

Example 28: You need to form a committee of three people. You have eight people to choose from. If one of them is named Bob, then how many different committees are possible?

Solution. Your committee might include Bob, or not. If you put Bob on the committee, then you have to choose two more people to put on the committee, from the seven remaining. This can be done in

$$C(7,2) = \frac{7!}{5!2!} = 21$$

ways.

If you don't put Bob on the committee, then you have to choose three more people on the committee from the seven remaining. You can do this in

$$C(7,3) = \frac{7!}{4!3!} = 35$$

ways. So the total number of ways is

$$C(7,2) + C(7,3) = 56.$$

Solution. The fact that one committee member is named Bob is a *red herring*: irrelevant to the solution. There are eight committee members total to choose from, and you need to choose any three of them, and so there are

$$C(8,3) = \frac{8!}{5!3!} = 56$$

ways.

By solving this problem twice – once, in a complicated way which used extra information – we saw an example of the Sum Formula for Combinations. The general Sum Formula works for the same reason.

Remark: You can also show the Sum Formula using algebra. The identity amounts to showing that

$$\frac{n!}{r!(n-r)!} = \frac{(n-1)!}{(r-1)!((n-1)-(r-1))!} + \frac{(n-1)!}{r!((n-1)-r)!},$$

which you can do by getting a common denominator on the right, and cleaning up all the messy algebra.

It is not too hard; try it if you want. But, in my opinion it's not all that interesting.

A multiplication rule for algebra. Suppose you want to multiply out $(x+y)^{10}$. Oooof! Sounds like a mess.

Let's try $(x+y)^3$ first. We have

$$(x+y)^3 = (x+y)(x+y)(x+y) = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy,$$

generalizing the 'FOIL' rule in high school algebra. But

$$xxy = xyx = yxx = x^2 y,$$

and so on, so that

$$(x+y)^3 = x^3 + 3x^2 y + 3xy^2 + y^3.$$

Similarly, we have

$$(x+y)^5 = x^5 + 5x^4 y + 10x^3 y^2 + 10x^2 y^3 + 5xy^4 + y^5.$$

How do we know? For example, to compute the coefficient of $x^3 y^2$, we count the total number of expressions with three x's and two y's. But, as we discussed before, there are exactly $C(5,2)$ of them!

This principle is very important, and it is called the *Binomial Theorem*.

Theorem 29 (The Binomial Theorem): We have

$$(x+y)^n = C(n,0)x^n + C(n,1)x^{n-1}y + \cdots + C(n,n)y^n.$$

Why is this so important? One reason is that it explains many of our other results. For example, if you substitute $x = y = 1$, you get

$$2^n = C(n,0) + C(n,1) + \cdots + C(n,n),$$

in other words the fact that the numbers in the nth row of Pascal's Triangle add up to 2^n. We saw this already, so this is another explanation.

What if we substitute $x = 1$ and $y = -1$? In that case we get

$$0 = C(n,0) - C(n,1) + C(n,2) - C(n,3) + \cdots \pm C(n,n),$$

where the last $\pm$ is a plus if n is even and a minus if n is odd. So, for example (with $n = 8$) we get

$$1 - 8 + 28 - 56 + 70 - 56 + 28 - 8 + 1 = 0,$$

which certainly isn't obvious by staring at it! This fact is at the heart of an advanced counting technique known as *inclusion-exclusion*.

Asymptotic behavior. Here is a computer simulation that allows you to play many, many rounds of Plinko – on a board with no walls, and which you can take to be large if you like.

Link: Plinko Simulation

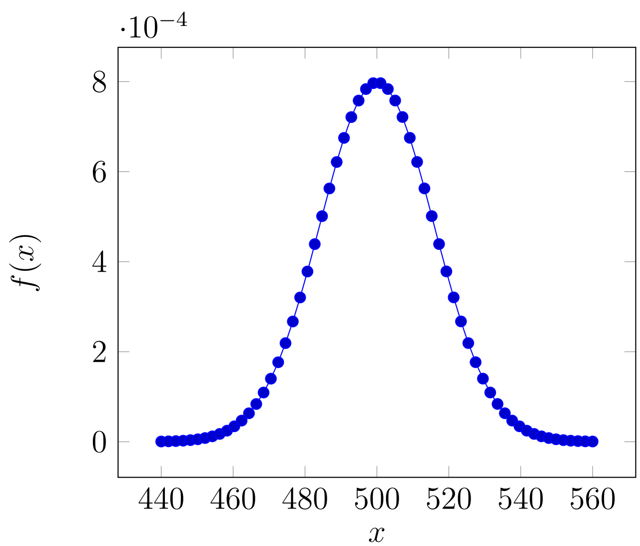
Perhaps you recognize the curve in the middle?

> **Theorem 30 (Central Limit Theorem):** As n goes to infinity, the distribution of the relative Plinko probabilities converges to a *bell curve*.

More specifically, it converges to a *normal distribution* with *mean* $\frac{n}{2}$ and *standard deviation* $\frac{\sqrt{n}}{2}$. For example, when $n = 1000$, this is the function

$$f(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1000} e^{-\frac{(x-500)^2}{500}},$$

which has the following graph:



Suppose you want to compute the probability that, if you flip 1000 coins, then between 460 and 480 of them are heads. Then this *equals* (up to a close approximation) *the area underneath this curve between* $x = 460$ *and* $x = 480$. You can see that you should not be too surprised if you get only 470 heads, but you should be very surprised if you get only 430.

Finally, if you enjoyed computing Plinko probabilities, you might enjoy this show, which combines elements of Plinko and Deal or No Deal, and adds its own twist:

Link: The Wall

There is much more we could say here, but we'll move on.

4.5 Exercises

1. Compute tables of $P(n, r)$ and $C(n, r)$ for all n and r with $0 \leq r \leq n \leq 8$.

2. Explain why $C(n, 0) = 1$ and $C(n, 1) = n$ for all n. Can you explain this using the definition instead of the formula?

3. You flip ten coins. What is the probability of the following outcomes?

   (a) Flipping all ten heads.

   (b) Flipping at least seven heads.

   (c) Flipping exactly five heads.

   (d) Flipping between three and seven heads.

4. You draw two poker cards. What is the probability that they are both –

   (a) spades?

   Solution. You can use the multiplication rule for probability for this. The probability the first is a spade is $\frac{13}{52}$, and the probability the second is a spade is $\frac{12}{51}$, so the answer is

$$\frac{13}{52} \times \frac{12}{51} = \frac{1}{17}.$$

   Here is an alternative solution. Consider the sample space consisting of all sets of two poker cards. Then we have

$$N(S) = C(52, 2) = 1326.$$

   If E is the event consisting of all sets of two spades, then we have

$$N(E) = C(13, 2) = 78,$$

   and

$$P(E) = \frac{78}{1326}.$$

   (b) of the same suit as each other?

   (c) adjacent in value (ace-two, two-three, etc., through king-ace)?

   Solution. As in the 'alternative solution' we have $N(S) = C(52, 2) = 1326$. So we have to count the number of pairs of cards which are adjacent in value to each other.

   To do this, first choose the higher card, any of the 52 cards in the deck is possible. Then, the rank of the lower card is fixed, so there are four possibilities (one in each suit). So $N(E) = 52 \times 4 = 208$, and

$$P(E) = \frac{208}{1326}.$$

(d) jacks or higher (including aces)?

(e) a pair?

(Note: There are multiple ways to solve these! Can you use combinations to count?)

5. In a game, you flip eight coins. Compute the expected value of the game if:

   (a) You get one dollar for each heads.

   (b) You get one dollar for each heads, and a ten dollar bonus if at least six land heads.

   (c) You get one dollar for each heads. If you flip fewer than four heads, then you have the opportunity to reflip all your coins once.

   If you do, you must reflip all your coins, not just those that landed tails. You have to accept the results of the second flip.

   (d) (Challenge) You get one dollar for each heads. If you flip fewer than four heads, then you have the opportunity to reflip all your coins, and you get to keep doing so until you flip at least four heads.

   (e) You get five dollars if you flip exactly five heads, and otherwise you get nothing.

6. Watch again the clip of Press Your Luck. Assume that the board has the same distribution of prizes as it did during Michael Larson's playing.

   Making some simplifying assumptions as necessary, formulate a strategy for the game. When should you press your luck, and when should you stop? (The answer should depend

7. You flip three coins. You then take any coins which landed tails and flip them a second time.

   (a) What is the probability that all land heads?

   (b) What is the probability that all land tails?

   (c) What is the expected number of heads?

8. The following clip is from the game show **Scrabble**:

   `https://www.youtube.com/watch?v=iliCKnHxJiQ`

   (a) At 6:25 in the video, Michael chooses two from eleven numbered tiles. The order in which he chooses them doesn't matter. Eight of the tiles are 'good', and reveal letters which are actually in the word. Three of them are 'stoppers'.
   How many different choices can he make?

   Solution. $C(11, 2) = 55$.
   In this example, it turns out that there are two R tiles, and two D tiles (one of which is a stopper). The easy solution presumes that these are different from each

other – that one of the numbered tiles is the R appearing first in the word, and another one is the R appearing second in the word.

However, if you watch the show a lot you will observe this is *not actually true* – the *first D* picked will always be the good one, and the second will always be the stopper. Our solution is the 'easy solution' – extra credit to anyone who observed that this is not quite accurate, and took account of it!

(b) Of these choices, how many choices don't contain a stopper? If he places both letters, what is the probability that both will actually appear in the word?

Solution. $C(8,2) = 28$, and so $\frac{28}{55}$.

(c) Michael can't guess the word and chooses two more of the remaining tiles. Now what is the probability that both of them will actually appear in the word?

Solution. Now there are nine remaining tiles and six of them are good. It's $\frac{C(6,2)}{C(9,2)} = \frac{15}{36} = \frac{5}{18}$. Not very good.

(d) At 8:15 (and for a different word), the contestants have used up two of the stoppers. Now what is the probability that both of Michael's letters will appear in the word?

Solution. There are six letters, and he chooses two. The probability that neither is the bad one is $\frac{4}{6} = \frac{2}{3}$.

(e) (Challenge!) Suppose that Michael knows the first (6:25) word from the beginning, but rather than guessing it immediately chooses to draw tiles until one of the following happens: (1) he draws and places the first R on the blue spot, and thereby can earn \$500 for his guess; (2) he draws two stoppers, and must play one of them (and so forfeits his turn); (3) he places all letters but the first R, and is obliged to guess without earning the \$500.

Compute the probabilities of each of these outcomes.

Solution. We look at this turn by turn.

- (First turn.) With probability $\frac{3}{55}$ he draws two stoppers and loses. With probability $\frac{10}{55} = \frac{2}{11}$ he draws the first R and can place it and win \$500.
  The number of ways in which he can draw one stopper and one good tile other than the first R is $3 \cdot 7 = 21$, so there is probability $\frac{21}{55}$ that this happens and he wins the turn but not \$500. Finally, there are $C(7,2) = 21$ in which he can draw good two tiles other than the first R, so there is probability $\frac{21}{55}$ that this will happen and he goes to a second round.
  Note that $3 + 10 + 21 + 21 = 55$ – a good way to check our work! We've enumerated all possibilities and the probabilities end up to 1.
- (Second turn.) There is probability $\frac{21}{55}$ that the game goes on to a second turn. The following probabilities assume that it does, and should all be multiplied by $\frac{21}{55}$.

There are $C(9,2) = 36$ ways to draw two tiles. As above, there are 3 ways to draw two stoppers, 8 ways in which he can draw the first R and something else, $3 \cdot 5 = 15$ ways in which he can draw a stopper and a tile other than the first R, and $C(5,2) = 10$ ways in which he can draw two more good tiles other than the first R. So there is probability $\frac{10}{36}$, or $\frac{21}{55} \times \frac{10}{36}$ total, of the game going onto a third round.

- (Third turn.) Similar to above. There are $C(7,2) = 21$ ways to draw two tiles, 3 to draw two stoppers, 6 to draw the first R, 9 to draw a stopper and a tile other than the first R, and 3 ways in which he can draw two more good tiles other than the first R. The probability of the game going on to a fourth turn (total) is $\frac{21}{55} \times \frac{10}{36} \times \frac{3}{21}$.
- (Fourth turn.) There are $C(5,2) = 10$ ways to draw two tiles, 3 to draw two stoppers, 4 to draw the first R, and 3 ways in which he can draw a stopper and a title other than the first R.

So we can compute all the probabilities:

- Places the first R:
$$\frac{10}{55} + \frac{21}{55} \cdot \frac{8}{36} + \frac{21}{55} \cdot \frac{10}{36} \cdot \frac{6}{21} + \frac{21}{55} \cdot \frac{10}{36} \cdot \frac{3}{21} \cdot \frac{4}{10} = \frac{10}{33}$$
- Draws two stoppers:
$$\frac{3}{55} + \frac{21}{55} \cdot \frac{3}{36} + \frac{21}{55} \cdot \frac{10}{36} \cdot \frac{3}{21} + \frac{21}{55} \cdot \frac{10}{36} \cdot \frac{3}{21} \cdot \frac{3}{10} = \frac{7}{66}$$
- Must guess without winning \$500:
$$\frac{21}{55} + \frac{21}{55} \cdot \frac{15}{36} + \frac{21}{55} \cdot \frac{10}{36} \cdot \frac{9}{21} + \frac{21}{55} \cdot \frac{10}{36} \cdot \frac{3}{21} \cdot \frac{3}{10} = \frac{13}{22}.$$

9. Consider our first model of Plinko, where we assumed that the puck would always go one space to the left or one space to the right, but did not ignore the walls of the board.

   (a) If the contestant drops the puck one slot to the left of center, we already computed the probability that the puck lands in each of the nine slots. Compute the expected value of this drop. (Use a calculator or computer, and round to the nearest dollar.)

   (b) Carry out all these computations (1) if the contestant drops the puck down the center, and (2) if the contestant drops the puck down the far left slot. If you have the patience, you might also do it if the contestant drops the puck two left of center – in this case, by symmetry, you will have considered all the possibilities.

   What can you conclude about where the contestant should drop the puck?

10. Watch one or more playings of Plinko, and discuss the shortcomings in our model. Does the puck ever go more than one space to the left or right?

    Briefly discuss how you would revise the model to be more accurate, and summarize how you would redo the problem above to correspond to your revised model. (The details are likely to be messy, so you're welcome to not carry them out.)

5 Poker

Note: This chapter will be not be covered, more than briefly, in the Spring 2018 course.

We digress from our discussion of 'traditional' game shows to discuss the game of *poker*. Is it a game show? Well, it is certainly a game, and you can most definitely find it on TV. Perhaps more to the point, it is an excellent source of interesting mathematical questions. The game is very mathematical, and we can very much use the mathematics we have developed to analyze it.

We start off by describing the poker hands from best to worst and solving the combinatorial problems which naturally arise. For example, if you are dealt five cards at random, what is the probability that you get dealt a straight? Two pair? A flush?

We then move on to discuss betting and the actual gameplay. Here is where expected value computations come into play: should you fold, call, or raise? To analyze these decisions, you must combine the mathematics with educated guesses about what cards your opponents might be holding. But beware that a skilled opponent will work to confound your guesses!

If you are interested in more, there are a variety of further resources available to you:

Online broadcasts. A lot of television broadcasts have found their way to the Internet. As of this writing, searching Youtube for 'poker tournament' or 'World Series of Poker' yields lots of hits. For example, here is the full final table (over five hours!) from the 2016 One Drop tournament:

Link: Poker – 2016 One Drop Tournament

Try to find broadcasts of 'full' final tables, where they show *all* the hands and not only the most 'interesting' ones. The edited versions show more big hands, big bets, and drama – and as such they offer a somewhat misleading perspective on the overall game.

Online play. It is possible to play poker online for free, without gambling. A site I have used myself is Replay Poker:

Link: Replay Poker

You play for 'chips', and betting is handled as usual, but the chips do not represent money.

Further reading. There are a great many excellent books on poker. I especially recommend the *Harrington on Hold'em* series by Dan Harrington and Bill Robertie. These books are quite sophisticated and walk you through a number of expected value and probability computations. If you've ever wanted to learn to play, you will find that this course provides excellent background!

5.1 Poker Hands

A **poker hand** consists of five playing cards. From best to worst, they are ranked as follows:

- **A straight flush**, five consecutive cards of the same suit, e.g. 5♠6♠7♠8♠9♠. An ace may be counted high or low but straights may not 'wrap around'. For example, AKQJT and 5432A both count as straights, but 432AK does not.

  If two players hold straight flushes, then the one with the highest high card counts as highest.

  As a special case, an ace-high straight flush is called a **royal flush**, the highest hand in poker. We will lump these in with straight flushes.

- **Four of a kind**, for example K♠K♣K♢K♡ and any other card. (If two players have four of a kind, the highest set of four cards win.)

- **A full house**, i.e. three of a kind and a pair, K♠K♣K♢7♡7♢. (If two players have a full house, the highest set of three cards wins.)

- **A flush**, any five cards of the same suit, e.g. Q♣10♣7♣6♣3♣. The high card breaks ties (followed by the second highest, etc.)

- **A straight**, any five consecutive cards, e.g. 8♣7♢6♢5♡4♠. The high card breaks ties.

- **Three of a kind**, e.g. 8♣8♢8♠A♡4♠.

- **Two pair**, e.g. 8♣8♢6♠6♡A♠.

- **One pair**, e.g. 8♣8♢6♠5♡A♠.

- **High card**, e.g. none of the above. The value of your hand is determined by the highest card in it; then, ties are settled by the second highest card, and so on.

We now compute *the probability of each possible hand occurring.* Our computations will make heavy use of the multiplication rule. (Note that each card is determined uniquely by its *rank* (e.g. king, six) and *suit* (e.g., spades, clubs).)

- **All hands**. The total number of possible hands is $C(52, 5) = 2598960$.

- **Straight flush** (including royal flush). There are four possible suits, and nine possible top cards of that suit: ace down through five. These determine the rest of the straight flush, so the total number of possibilities is $4 \times 10 = 40$.

- **Four of a kind**. There are thirteen possible ranks. You must hold all four cards of that suit, and then one of the other 48 cards in the deck, so the total number of possibilities is $13 \times 48 = 624$.

- **Full house**. First, choose the rank in which you have three of a kind. There are 13 possible ranks, and $C(4, 3) = 4$ choices of three of that rank. Then, choose another rank (12 choices) and two cards ($C(4, 2) = 6$) of that rank. The total number of possibilities is the product of all these numbers: $13 \times 4 \times 12 \times 6 = 3744$.

- **Flush**. Choose one of four suits (in 4 ways), and five cards of that suit (in $C(13,5)$ ways), for a total of $4 \times C(13,5) = 5148$ possibilities.

  Except, we don't want to count the straight flushes again! So subtract 40 to get 5108.

- **Straight**. Choose the highest card (ace through five, so ten possibilities). For each of five ranks in the straight, there are 4 cards of that rank, so the number of possibilities is $10 \times 4^5 = 10240$. Again subtracting off the straight flushes, we get 10200.

- **Three of a kind.** Choose a rank and three cards of that rank in $13 \times C(4,3) = 52$ ways. Then, choose two other ranks (distinct from each other) in $C(12,2)$ ways. For each of these ranks there are four possibilities, so the total is $52 \times C(12,2) \times 4^2 = 54912$.

  Note that hands with four of a kind or a full house 'include three of a kind', but we counted so as to exclude these possibilities, so we don't need to subtract them now.

- **Two pair.** Choose two different ranks in $C(13,2)$ ways; for each, choose two cards of that rank in $C(4,2)$ ways. Finally, choose one of the 44 cards not of the two ranks you chose. The total number of possibilities is $C(13,2) \times C(4,2)^2 \times 44 = 123552$.

- **One pair.** Choose the rank in 13 ways and choose two cards of that rank in $C(4,2)$ ways. Then, choose three other ranks in $C(12,3)$ ways and for each choose a card of that rank in 4 ways.

  The total number of possibilities is $13 \times C(4,2) \times C(12,3) \times 4^3 = 1098240$ ways.

- **None of the above.** There are several ways we could count this. Here is one way: we can choose five different ranks in $C(13,5)$ ways – but we must subtract the ten choices that are straights. So the number of choices for ranks is $(C(13,5) - 10)$.

  Now, for each rank, we choose a suit, and the total number of choices is $4^5 - 4$. We subtract 4 because we want to exclude the flushes! So the total number of possibilities is $(C(13,5) - 10) \times (4^5 - 4) = 1302540$.

  Here is a second way to get the same result. We know that the total number of possibilities is 2598960. So we add all the previous possibilities, and subtract from 2598960.

This involved some subtleties, and for other variations the computations are still harder! For example, in **seven card stud** you are dealt a seven-card hand, and you choose your best five cards and make the best possible poker hand from these. You can redo all the above computations, but now some new possibilities emerge. For example, you can be simultaneously dealt a straight and three of a kind – and you want to count this only as a straight (since that is better than three of a kind). But it is not *so* hard. The following Wikipedia page works out all the probabilities in detail:

`https://en.wikipedia.org/wiki/Poker_probability`

Poker variations. There are many variants of poker. The rules for betting (and blinds and antes) are described in the next section; for now we simply indicate when a round of betting occurs.

'Ordinary' poker. (No one actually plays this.) Each player is dealt five cards face down. There is a round of betting. The best hand (among those who have not folded) wins.

Five-card draw. Each player is dealt five cards face down. There is a round of betting. Then, each player who has not folded may choose to trade in up to three cards, which are replaced with new cards (again dealt face down). There is another round of betting, and the best hand wins.

Texas Hold'em. Typically played using blinds (and sometimes also antes), applied to the first round of betting only. Each player is dealt two cards, dealt face down. There is a round of betting. Three community cards are dealt face up (the 'flop'), which every player can use as part of their hand. There is a round of betting. A fourth community card is dealt (the 'turn'), followed by another round of betting. Finally, a fifth community card is dealt (the 'river'), again followed by another round of betting.

Each player (who has not folded) chooses their best possible five-card hand from their two face-down cards and the five face-up cards (the latter of which are shared by all players). The best hand wins.

Texas Hold'em is extremely popular and plenty of video can be found on the internet. For example, this (six hour!) video is of the first part of the final table of the 2014 World Series of Poker:

`https://www.youtube.com/watch?v=5wlVFMNVJZQ`

The top prize was a cool $10 million.

This is the most interesting poker video I have ever seen. Most telecasts of poker heavily edit their coverage, only showing the hands where something exciting or out of the ordinary happens. This video is unedited, and so gives a much more realistic viewpoint of what tournament poker is like.

In the opening round of Texas Hold'em, you are dealt only your two-card hand and you have to bet before any of the community cards are dealt. This offers some probability questions which are quite interesting, and easier than those above. For example, in *Harrington on Hold'em, Volume I: Strategic Play*, Harrington gives the following advice for you should raise, assuming you are playing at a full table of nine or ten players and are the first player to act.

- Early (first or second) position: Raise with any pair from aces down to tens, ace-king (suited or unsuited), or ace-queen (suited).

- Middle (third through sixth) position: Raise with the above hands, nines, eights, ace-queen, ace-jack, or king-queen (suited or unsuited).

- Late (seventh or eighth) position: Raise with all the above hands, sevens, ace-x, or high suited connectors like queen-jack or jack-ten.

Harrington also points out that your strategy should depend on your stack size, the other players' stack sizes, your table image, the other players' playing styles, any physical tells you have on the other players, the tournament status, and the phase of the moon. But this is his starting point. Let us work out a few examples (you will be asked to work out more in the exercises).

Example 5.1 *In a game of Texas Hold'em, compute the probability that you are dealt a pair of aces ('pocket aces').*

Solution. There are $C(52, 2) = 1326$ possible two-card hands. Of these, $C(4, 2) = 6$ are a pair of aces, so the answer is $\frac{6}{1326} = \frac{1}{221}$, a little bit less than 0.5%.

Example 5.2 *In a game of Texas Hold'em, compute the probability that you are dealt a pair.*

Solution. There are 13 possible ranks for a pair, and $C(4, 2) = 6$ pairs of each rank, so the answer is $\frac{6\times 13}{1326} = \frac{1}{17}$.

Example 5.3 *You are playing Texas Hold'em against five opponents, and you are dealt a pair of kings. You have the best hand at the table unless someone else has a pair of aces. Compute the probability that one of your opponents has a pair of aces.*

Approximate solution. There are fifty cards left in the deck, excluding your two kings. The probability that any *specific* one of your opponents has pocket aces is $\frac{C(4,2)}{C(50,2)} = \frac{6}{1225}$, or about 1 in 200. (This much is exact.)

These probabilities are not independent: if one player has pocket aces, the others are less likely to. Nevertheless, we get a very nearly correct answer if we assume they are independent. The probability that any specific player does *not* have pocket aces is $1 - \frac{6}{1225} = \frac{1219}{1225}$. If these probabilities are independent, the probability that all five opponents have something other than pocket aces is $\left(\frac{1219}{1225}\right)^5$. So the probability that at least one of your opponents has pocket aces is

$$1 - \left(\frac{1219}{1225}\right)^5 = 0.0242510680\ldots$$

Remark. Here is a simpler approximate solution. Just multiply $\frac{6}{1225}$ by 5, to get

$$\frac{30}{1225} = 0.02448979591\ldots$$

This is almost exactly the same. Why is this? We can use the binomial theorem to see that

$$1 - (1 - x)^5 = 5x - 10x^2 + 10x^3 - 5x^4 + x^5,$$

and plug in $x = \frac{6}{1225}$. Since x is very small, the x^2, etc. terms are **very** small.

Example 5.4 *You are sitting in first position. Compute the probability that you receive a hand that you should raise, according to Harrington's advice.*

Solution. As before there are 1326 hands, so we count the various hands that Harrington says are worth opening:

- A pair of aces through tens: Five ranks, and 6 ways to make each pair, so a total of $5 \times 6 = 30$.
- Ace-king: Four ways to choose the suit of the ace, and four ways to choose the suit of the king. $4 \times 4 = 16$.
- Ace-queen suited. (*Suited* means the cards are of the same suit. If your cards are suited, this helps you because it increases the chances that you will make a flush.) Four ways to choose the suit, so just 4.

None of these possibilities overlap, so the total number is $30 + 16 + 4 = 50$. The probability is $\frac{50}{1326}$.

This is less than 1 in 25! Harrington's strategy is much more conservative than that of most top players.

In the exercises, you will compute the probability of getting a hand worth opening in middle or late position.

5.2 Poker Betting

So far we have just considered probabilities. But the interesting part of the game comes when we combine this with a discussion of betting strategy.

Poker is played for *chips*, which may or may not represent money. In general there are two different formats. In a **cash game**, you simply try to win as many chips as you can. By contrast, a **tournament** is played until one player has won all the chips. Before each hand players have to put **antes** or **blind bets** into the pot, and in a tournament these keep going up and up to force the tournament to end eventually.

Betting rounds. In all variations of poker, a betting round works as follows. The first player (usually, but not always, the player left of the dealer) opens the betting. She may **check** (bet nothing) or bet any amount. The betting then proceeds around the table clockwise. If no one has bet yet, the player may check or bet. If someone has bet, then the player may **fold** (abandon her hand), **call** (match the bet), or **raise** (put in a larger bet). The betting continues to go around the table until either everyone has checked, or everyone has called or folded to the last (largest) bet. Note that players may raise an unlimited number of times, so betting can go around the table multiple times if many players keep raising.

In *no-limit* poker, a player may bet anything up to and including her entire stack of chips. Players are never allowed to bet more than however many chips they have on the table. (You are not allowed to reach into your wallet and suddenly drop a stack of Benjamins.)

Conversely, you can always call a bet for your entire stack: if someone bets more chips than you have, you may go 'all-in' and their effective bet is limited to the number of chips you have. (There are 'side pot' rules if one player is all-in and two other players want to keep raising each other; we won't consider them here.)

Typically there are multiple rounds of betting. If a player bets and everyone else folds, then that player wins the pot. (The 'pot' consists of the blinds and antes and all of the bets that have been made.) Otherwise, everyone remaining at the end compares their hands, and the best hand wins the pot.

Blinds and antes. A hand of poker never starts with an empty pot; there is always a little bit of money to be won from the beginning. This is assured via blinds and antes. If **antes** are used, then each player puts a fixed (small) amount of money into the pot at the beginning. If **blinds** are used, then the first two players in the first betting round make a 'blind bet' before looking at their cards. For example, the first player might be required to bet \$1 (the small blind) and the second player \$2 (the big blind). These count as their initial bets, except that if everyone calls or folds to the big blind, the round is not quite over; the big blind has the opportunity to raise if she wishes.

5.3 Examples

We now consider some examples of poker play and the mathematics behind your decision making.

Example 1. You are playing Texas Hold'em with one opponent (Alice). The current pot is 500 chips, and you and Alice each have 700 chips. You have a hand of $5\heartsuit 4\heartsuit$, the flop comes $A\clubsuit K\heartsuit 10\heartsuit$. You check, and Alice responds by going all-in. Should you fold or call her bet?

Analysis. There are three steps to solving this problem. First, you estimate your winning probability depending on what cards come. Since you don't know what your opponent has, this is a very inexact science (and indeed depends on your assessment of Alice's strategy).

The next two steps are mathematically more straightforward: the second step is to compute the probability of each possible outcome, and the third is to determine whether the expected value of calling is positive or negative. Since the expected value of folding is always zero (not counting whatever you have put into the pot already), this determines whether or not you should call.

You guess that Alice probably has a good hand – a pair of tens or higher. You estimate that you probably need to make a flush to beat her. You make a flush if at least one heart comes in the turn and the river. You'd rather see *only* one heart, because if two hearts come, Alice beats you if she has any heart higher than the $5\heartsuit$.

- If exactly one heart comes during the next two cards, then almost certainly you win. You only lose if Alice has two hearts, one of them higher than a five, or if she makes some freak hand like a full house or four of a kind. (This can't be discounted if a pair appears on the flop, but as it stands this looks pretty unlikely.)

We estimate your winning chances here as 90%. (Reasonable people might disagree!)

- If two hearts come during the next two cards, you might win – but Alice could easily have a heart higher than the 5♡. We estimate your chances of winning as 50%.
- If no hearts come, then you are very unlikely to win. You could – for example, if two fives, or two fours, or a five and a four, come then you *might* win, but this is unlikely. We will simplify by rounding this probability down to zero.

There are 47 cards you can't see, and nine of them are hearts. What is the probability that the next two are both hearts? As we've seen before, this is

$$\frac{9}{47} \cdot \frac{8}{46} \sim 0.033\ldots$$

This is quite low! It is substantially lower than $(1/4)^2$, simply because you can already see four of the hearts.

Now, what is the probability that one, but not both, of the next two cards, is a heart? There are two ways to compute this, and we will work out both.

Method 1. The probability that the first card is a heart and the second card is not a heart is

$$\frac{9}{47} \cdot \frac{38}{46} \sim 0.158\ldots$$

The probability that the second card is a heart and the first card is not is the same. So the total probability is $\frac{342}{1081}$, or approximately 0.316.

Method 2. First, we compute the probability that neither card is a heart. This is

$$\frac{38}{47} \cdot \frac{37}{46}.$$

So, the probability that exactly one card is a heart is

$$1 - \frac{38}{47} \cdot \frac{37}{46} - \frac{9}{47} \cdot \frac{8}{46} = \frac{342}{1081}.$$

It is very typical that there are multiple ways to work out problems like this! This offers you a great chance to check your work.

So what's the probability you win? 0.9 times the probability that exactly one heart comes, plus 0.5 times the probability that two hearts come. In other words,

$$0.9 \times 0.316 + 0.5 \times 0.033 \sim 0.301,$$

which for the sake of simplicity we will round off to 0.3.

Now, on to the expected value computation. If you call and win, then you win \$1,200: the \$500 previously in the pot, plus the \$700 that Alice bet. If you call and lose, you lose \$700. Therefore the expected value of calling is

$$0.3 \cdot 1200 + 0.7 \cdot (-700) = -130.$$

It's negative, so you should fold here.

But notice that it's close! So, for example, if the flop had come $A\heartsuit 8\heartsuit 7\clubsuit$, then you should call. (Exercise: verify this as above!) Here you will make a straight if a six comes. It is not so likely that a six will come, but a small probability is enough to swing your computation.

Example 2. The same situation, except imagine that you both have 1,000 chips remaining and that Alice bets only 300 chips. What should you do?

You could consider folding, calling, or now raising. Let us eliminate raising as a possibilty: if Alice is bluffing with something like $Q\clubsuit 7\diamondsuit$, then you might get her to fold, even though she has a better hand. But this doesn't seem very likely.

Since you have the opportunity to bet again, let us now consider **the next card only.**

- Suppose the next card is a heart, giving you a flush. Then, you think it is more likely than not that you'll win, so you want to bet. Moreover, since Alice might have one heart in her hand, you would really like her to fold – and so if this happens, you will go all in.

  It is difficult to estimate the probabilities of what happens next – this depends on how you see Alice, how she sees you, and what she's holding. As a rough estimate, let us say there is a 50-50 chance that she calls your all-in bet, and if she calls there is a 75% chance of you winning with your flush.

- Suppose the next card is not a heart. Then you don't want to bet, because you don't have anything. Let us say that there is a 75% chance that Alice goes all-in, in which case you should and will fold. (Check the math here!)

  If Alice instead checks (assume there is a 25% chance of this), you both get to see one more card and bet again. If it is a heart, assume that you both go all-in and that you win with 75% probability. If it is not a heart, assume that Alice goes all in and you fold.

These percentages are approximate – once again we can't really expect to work exactly. But given the above, we can enumerate all the possibilities, their probabilities, and how much you win or lose:

- Heart, she calls your all-in, you win: probability $\frac{9}{47} \times \frac{1}{2} \times \frac{3}{4} \sim 0.072$, you win \$1500. (The initial \$500 pot, and her \$1000.)

- Heart, she calls your all-in, you lose: probability $\frac{9}{47} \times \frac{1}{2} \times \frac{1}{4} \sim 0.024$, you lose \$1000. (Your remaining \$1000.)

- Heart, she folds: $\frac{9}{47} \times \frac{1}{2} \sim 0.096$, you win \$800. (The initial \$500 pot, plus the \$300 she invested to make the first bet.)

- Not a heart, she goes all-in: $\frac{38}{47} \times \frac{3}{4} \sim 0.606$, you lose \$300. (This is what you invested to call her first bet, but you fold and so avoid losing any more.)

- Not a heart, she checks, next card is a heart, you win: $\frac{38}{47} \times \frac{1}{4} \times \frac{9}{46} \times \frac{3}{4} \sim 0.030$. You win \$1500.

- Not a heart, she checks, next card is a heart, you lose: $\frac{38}{47} \times \frac{1}{4} \times \frac{9}{46} \times \frac{1}{4} \sim 0.010$. You lose \$1000.

- Not a heart, she checks, next card is not a heart, you fold: $\frac{38}{47} \times \frac{1}{4} \times \frac{37}{46} \sim 0.163$. You lose \$300.

As is often the case in poker, it is more likely that you will lose than win, but the winning amounts are larger than the losing amounts. Here there are two reasons for this: first of all, if she goes all-in on a bad card for you, then you can usually fold and cut your losses. The second is that we're comparing against a baseline of folding, which we say has expected value zero. But if you bet, you can not only get Alice to match your bets, but also keep your stake in the existing pot.

The expected value of calling is

$$.072 \times 1500 - .024 \times 1000 + .096 \times 800 - .606 \times 300 + .040 \times 1500 - .013 \times 1000 - .163 \times 300 \sim -35.$$

A close decision, but if we believe our assumptions, then it looks like it's wise to fold.

Example 3. You are the big blind (50 chips) at a full table, playing Texas Hold'em. The first player, who is known to be conservative, raises to 200 chips, and everyone else folds to you. You have a pair of threes, and if you call, both you and your opponent will have 3,000 more chips to bet with. Since you already have 50 chips in the pot, it costs you 150 chips to call.

Should you call or fold?

To solve this problem we again have to make guesses about what we think will happen, which are still more inexact than the last problem. This will set up another expected value problem.

Anyway, the first player is known to be conservative, so she probably has ace-king or a high pair or something like that. Let us assume that *no three comes on the flop, you will not dare to bet.* Assume further that your opponent will, and you end up folding.

Since you have a pair of threes, you are hoping that a three comes on the flop. If so, you will almost certainly win. Let us assume that, if a three comes on the flop:

- With 25% probability, your opponent will fold immediately and you will win the current pot (of 425 chips: your bet, her bet, and 25 chips from the small blind).

- With 60% probability, your opponent will bet somewhat aggressively, but eventually fold, and you win (on average) the current pot of 425 chips, plus an additional 500 chips.

- With 10% probability, your opponent will bet very aggressively. Both of you go all-in, and you win the pot of 425 chips plus all 3,000 of her remaining chips.

- With 5% probability, your opponent gets a better hand than three threes, and both of you go all-in and you lose 3,000 of your remaining chips.

Let α be the probability of a three coming on the flop. Then, the expected value of calling (relative to folding) is

$$-150 + \alpha \cdot \Big(.25 \cdot 425 + .60 \cdot 925 + .10 \cdot 3425 + .05 \cdot (-3000) \Big) = -150 + 853.75\alpha.$$

So we need to compute α to determine whether this is positive or negative. To illustrate our techniques, we will do this in two different ways. In both cases we compute the probability that **no** three comes on the flop, and then subtract this from 1.

Solution 1. The first card will not be a three with probability $\frac{48}{50}$: there are 50 cards remaining, and 48 of them are not threes. If the first card is not a three, then the second card will not be a three with probability $\frac{47}{49}$, and the third card will not be a three with probability $\frac{46}{48}$. The probability that at least one card is a three is therefore

$$1 - \frac{48}{50} \cdot \frac{47}{49} \cdot \frac{46}{48} = .117\cdots$$

Therefore, the expected value of calling is

$$-150 + 835.75 \cdot .117 = -52.30.$$

It is negative, so a call is more prudent.

Solution 2. We compute in a different way the probability that none of the three cards in the flop is a three. There are $C(50,3)$ possible flops, and $C(48,3)$ possible flops which don't contain a three. So this probability is $\frac{C(48,3)}{C(50,3)}$, which is the same as $\frac{48}{50} \cdot \frac{47}{49} \cdot \frac{46}{48}$.

Some remarks:

- If you each had 10,000 remaining chips, then it **would** make sense to call. (Redo the math to see why!!) This illustrates the principle that long-shot bets are more profitable if you possibly stand to make a very large amount of money.

- The above computations assumed that all 50 cards were equally probable. But, given what you know about your opponent, you might assume that she doesn't have a three in her hand. In this case, the probability of getting a three on the flop goes up to

$$1 - \frac{46}{48} \cdot \frac{45}{47} \cdot \frac{44}{46} = .122\cdots$$

  which is slightly higher.

5.4 Exercises

Thanks to the participants (credited by their screen names below) in the Two Plus Two Forums for suggesting poker hands which are treated here:

`http://forumserver.twoplustwo.com/32/beginners-questions/videos-online-illustrating-simple-mathematical-poker-concepts-1631031/`

1. Refer to Harrington's opening strategies for Texas Hold'em described above. If you are in middle position and everyone has folded before you, compute the probability that you are dealt a hand which Harrington suggests raising.

   Now do the same for late position.

2. (Suggested by ArtyMcFly.) The following amusing clip shows a hand in a million-dollar Hold'em tournament with eight players, where two players are each dealt a pair of aces. One of them makes a flush and wins.

   `https://www.youtube.com/watch?v=aR52zv1GqBY`

   (a) Compute the probability that Drinan and Katz are each dealt a pair of aces. (No need to approximate; you can compute this exactly.)

   (b) Compute the probability that any two of the eight players are each dealt a pair of aces.

   (c) Given that two players are dealt aces, these aces must be of different suits. Each player will win if at least four cards of one of his two suits are dealt. (If four of this suit are dealt, then he will make a flush. If five of this suit are dealt, then both players will have a flush, but only one of them will have an ace-high flush.)

   The broadcast lists a probability of 2% of this happening for each player. Compute this probability exactly.

   (Note that the most common outcome is that no four cards of the same suit will be dealt, in which case the two players will have equal hands and tie.)

   (d) Compute the probability of this whole sequence happening: two of the eight players are dealt a pair of aces, and one of them makes a flush and wins. Please give both an exact answer and a decimal approximation.

   (e) Suppose these eight players play one hundred hands of poker. What is the probability that this crazy sequence of events happens at least once?

3. (Suggested by whosnext.) Here is another clip illustrating some serious good luck. (Or bad luck, depending on whose perspective you consider!)

   `https://www.youtube.com/watch?v=72uxvL8xJXQ`

Danny Nguyen is all-in with $A\diamondsuit 7\diamondsuit$ against an opponent with $A\spadesuit K\clubsuit$. The flop is $5\heartsuit K\heartsuit 5\spadesuit$. After this, the next two cards must both be sevens for Nguyen to win. Compute the probability of this happening.

(Note: there is also a small possibility of a tie, for example if both cards are aces.)

4. Consider a variant of poker where you are dealt four cards instead of five. So a 'straight' consists of four consecutive cards, a 'flush' four of a suit.

   By analogy with ordinary poker, determine what the possible hands are, and determine the probability of each. For each hand, give an exact answer for the probability as well as a decimal approximation.

5. (This is Hand 4-3 from *Harrington on Hold'em, Volume 1.*)

   Early in a poker tournament, with blinds \$5 and \$10, you are sitting third out of ten players in a no-limit Hold'em tournament with a stack of \$1,000. You are dealt $A\diamondsuit K\spadesuit$.

   The first two players fold, and you elect to raise to \$50. The next four players fold, and the eighth (next) player, who has a stack of \$1,630, calls your bet. The total pot is \$115, and the remaining players fold.

   The flop comes $J\diamondsuit 7\clubsuit 4\heartsuit$, and you act first. You choose to bet \$80. (This is a 'continuation bet', a kind of bluff. Since you expect that your opponent is somewhat likely to fold, this is considered good strategy.)

   Your opponent raises to \$160. **Do you fold, call, or raise the bet?**

   You should analyze this hand as in the examples in the book and in lecture. As best as you can, estimate your odds of having the best hand after the turn and the river, and carry out an appropriate expected value computation.

   Note: There is no single right answer, so justify your assumptions. If you like, you may work with one other person in the class and turn in a joint soution to this problem.

6. (This is the optional bonus.) Watch part of the World Series of Poker clip in the text, or any other poker tournament which is publicly available. (With your solution, please let me know where I can find video to watch the hand myself.) Find a decision made by one of the players similar to the situation in the text or the previous problem, and either explain or critique the play. Your solution should involve probability and expected value computations somewhere!

6 Inference

Your instructor decides to conduct a simple experiment. He pulls out a coin and is curious to see how many consecutive heads he will flip. He starts flipping – and lo and behold he flips a long sequence of consecutive heads! Six, seven, eight, nine, ten .... What are the odds of consecutive heads? $\left(\frac{1}{2}\right)^{10} = \frac{1}{1024}$. Pretty unlikely!

He continues flipping. Eleven, twelve, thirteen, fourteen, ... the probabilities get smaller and smaller. But eventually it occurs to you that there is an alternative, and indeed more likely, explanation: *You cannot see the coins*, and so *perhaps your instructor was just lying to you.*

What happened here? After the first coin, or after the second, you probably didn't suspect any dishonesty – after all, it is not so unlikely to flip one or two heads. He *could have been* lying, but you probably didn't suspect that. But while the probability of umpteen consecutive heads goes down and down, the probability that he was lying from the beginning doesn't, and eventually the latter becomes more plausible.

This is an example of *Bayesian inference*, which we will explore from a mathematical point of view. But even if you don't know the mathematics yet, you already make similar inferences all the time. For example, suppose that a politician makes a claim you find surprising.[3] Then, informally you will assess the probability that the claim is true. In doing so, you will take into account two factors: (1) how likely you believed this claim might have been true, before the politician made it; (2) your assessment of the honesty of the politician in question.

And finally we can look for examples from game shows. Here is a clip of *Let's Make a Deal*:

Link: Let's Make a Deal

What would you do?

6.1 Conditional Probability

[3]More specifically, this claim should concern a *matter of fact*, which can be independently verified to be true or false. For example, a politican might claim that crime levels have been rising or falling, that the moon landing was faked, or that Godzilla was recently sighted in eastern Siberia. Even if such claims cannot be confirmed or denied with 100% accuracy, the point is that they are objectively true or false. This is different than offering an opinion or speculation. For example, a politician might claim that if we airlift ten million teddy bears into North Korea, they will overthrow their dictator and become a democracy. We cannot say this is true or false without trying it. Similarly, a politician might say that Americans are the kindest people in the world. Unless you are prepared to objectively measure 'kindness', this is a subjective matter of opinion.

Definition 1 (Conditional Probability): Let A and B be events in a sample space S. If $P(A) \neq 0$, then the **conditional probability of** B **given** A, written $P(B|A)$, is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

Here the symbol $\cap$ means **intersection** – $A \cap B$ is the set of outcomes that are in both A and B. In other words, $P(A \cap B)$ is the probability that both A and B occur.

We also sometimes omit the word 'conditional', and just say 'the probability of B given A'.

Example 2: You flip two coins. Compute the probability that you flip at least two heads, given that you flip at least one head.

Solution. For clarity's sake, we will do everything 'the long way' and not avail ourselves of any shortcuts.

The sample space is

$$S = \{HH, HT, TH, TT\},$$

with all outcomes equally likely. Write A for the event that we flip at least one head, and B for the event that we flip two heads. We have

$$A = \{HH, HT, TH\},$$

$$B = \{HH\},$$

and

$$A \cap B = B = \{HH\}.$$

So,

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

Remark: In the example above, we had $B = A \cap B$ because B is a subset of A. The next example will illustrate a problem where this isn't the case.

Remark: An alternative formula for expected value is

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{N(A \cap B)}{N(S)}}{\frac{N(A)}{N(S)}} = \frac{N(A \cap B)}{N(A)},$$

which is sometimes easier to use. This is because $P(A \cap B) = \frac{N(A \cap B)}{N(S)}$ and $P(A) = \frac{N(A)}{N(S)}$, so that when you compute $P(B \mid A)$ the denominators cancel.

Example 3: You roll two dice. What is the probability that the sum of the numbers showing face up is 8, given that both dice show an even number?

Solution. Writing S for the sample space; it has 36 elements as we have seen before. Write A for the event that the numbers are both even, and B for the event that the total is eight. Then we have

$$A = \{22, 24, 26, 42, 44, 46, 62, 64, 66\},$$

$$B = \{26, 35, 44, 53, 62\},$$
$$A \cap B = \{26, 44, 62\}.$$

Then (using the alternate version of our formula) we have

$$P(B|A) = \frac{N(A \cap B)}{N(A)} = \frac{3}{9} = \frac{1}{3}.$$

In conclusion, if we know that both dice show an even number, then the total is more likely to be eight. This is true even though we removed some possibilities like $3 + 5$.

We'll now use this to analyze the strategy of the Price Is Right Game **Hot Seat**. Here is a clip:

Link: The Price Is Right – Hot Seat

Game Description (Hot Seat (The Price Is Right)): The game is played for a cash prize of up to $20,000.

For each of five small prizes, the contestant is shown a price and is asked whether the actual price is higher or lower. She can win an increasing amount of money based on how many prizes she has correctly priced:

Correct Answers	Payoff
1	500
2	2500
3	5000
4	10000
5	20000

Once she has made a guess for each of the prizes, her seat is moved to one prize at a time: first, to each of the prizes which she has guessed correctly (in random order), and then to a prize which she has guessed incorrectly (chosen at random).

At each stage, she is asked whether she wants to end the game and keep whatever she has won, or keep going. If she keeps going, and she has guessed the prize in front of her correctly, then she moves to the next higher price level and the game continues. If she keeps going and has guessed wrong, she leaves with nothing.

To analyze the game, we will *assume that all of her guesses are random and she has no idea whether any of them are correct.* (After you read through this analysis, you might consider what would change if this assumption is changed.)

Then, here are the probabilities for the number of correct answers:

Correct Answers	Probability
0	$\frac{1}{32}$
1	$\frac{5}{32}$
2	$\frac{10}{32}$
3	$\frac{10}{32}$
4	$\frac{5}{32}$
5	$\frac{1}{32}$

This is the same computation as for Plinko probabilities or flipping coins: the probability that she has given exactly n correct answers is $C(5, n)/32$.

We now rewrite the table: for each n, here are the probabilities that she has given *at least* n correct answers:

Correct Answers	Probability
≥ 1	$\frac{31}{32}$
≥ 2	$\frac{26}{32}$
≥ 3	$\frac{16}{32}$
≥ 4	$\frac{6}{32}$
≥ 5	$\frac{1}{32}$

In the video clip, the contestant has guessed three prices correctly and advanced to the $5,000 level, and decides to keep her money and walk. Was is the right decision?

This is a conditional probability exercise. We need to compute the probability that she has guessed four prices correctly, given the information that she has guessed three correctly.

Let B be the event that she has guessed four or more prices correctly, and let A be the event that she has guessed three or more prices correctly.

The *prior* probability that she has answered at least four correctly is $\frac{6}{32}$; that is, we have

$$P(B) = P(A \cap B) = \frac{6}{32}.$$

The probability that she has answered at least three correctly is

$$P(A) = \frac{16}{32}.$$

Therefore, we have

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{6}{32}}{\frac{16}{32}} = \frac{6}{16} = \frac{3}{8}.$$

In other words, once she has advanced to the fourth seat, the probability that she has guessed that price correctly is $\frac{3}{8}$ (given our assumptions). Her decision to walk agrees with the math!

Remark: The *prior*, or *unconditional* probability of an event is the probability of that event occurring, when the relevant additional evidence is *not* taken into account.

In our conditional probability formula, the prior probability of the event B is $P(B)$, the 'additional evidence' is that the event A has occurred, and the conditional probability is $P(B \mid A)$.

6.2 The Monty Hall Problem

We come at last to the most famous game show math question in history: the *Monty Hall Problem*, inspired by the show *Let's Make a Deal*. (Monty Hall was the name of its longtime host.)

You saw a typical clip from Let's Make a Deal above. Unfortunately, the scenario never actually happened on the show. It was apparently first posed by Steve Selvin in a 1975 letter to *American Statistician*, and then popularized by Marilyn vos Savant in *Parade Magazine*.

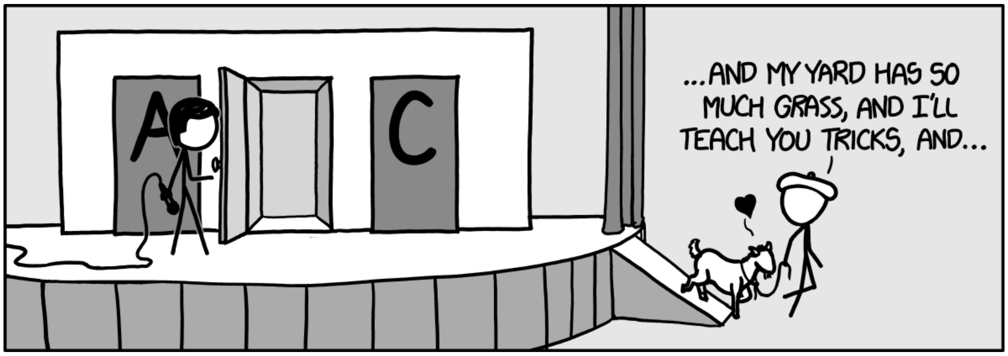
Since there is no clip to show, we have to describe it instead:

The Monty Hall Problem: Monty Hall shows you three doors. Behind one door is a car, and behind the others are goats.

You pick a door, say No. 1. The host, who knows what's behind the doors, opens another door, say No. 3, behind which is a goat. He then asks you if you want to switch your guess to Door No. 2. Should you?

We'll have to make several assumptions. The first is that you're not this[4] person:

[4] Comic strip credit: xkcd, 'Monty Hall', by Randall Monroe. `https://xkcd.com/1282/`.



We will make the following further assumptions:

- Initially, the car is equally likely to be behind any of the three doors.
- After you choose a door, the host will randomly pick one of the other doors with a goat and open that one.

  More specifically: If you choose a door with a goat, then exactly one of the other two doors will have a goat and the host will show it to you. If you choose the door with the car, then both of the other doors will have goats and the host will pick one of them at random and show it to you.

So, given that you choose Door 1, let's compute the sample space of all possible outcomes:

- The car is behind Door 2 (probability $\frac{1}{3}$). Monty shows you Door 3.
- The car is behind Door 3 (probability $\frac{1}{3}$). Monty shows you Door 2.
- The car is behind Door 1, and Monty shows you Door 2. (Probability $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$).
- The car is behind Door 1, and Monty shows you Door 3. (Probability $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$).

Let B be the event that the car is behind Door 2 (so $P(B) = \frac{1}{3}$), and let A be the event that Monty shows you Door 3. We want to compute $P(B|A)$, the probability that the car is behind Door 2, given that Monty showed you Door 3.

We have

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

The probability $P(A \cap B)$ is $\frac{1}{3}$, the same as $P(B)$. As we saw before, if the car is behind Door 2, Monty will always show you Door 3.

The probability $P(A)$ is $\frac{1}{2}$, the sum of the two probabilities above in which Monty shows you $\frac{1}{2}$. If the car is behind Door 2, Monty will always show you Door 3, and if the car is behind Door 1 then Monty *might* show you Door 3.

So

$$P(B|A) = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}.$$

Given that Monty showed you Door 3, there is now a $\frac{2}{3}$ probability the car is behind Door 2. You should switch.

We will discuss the problem further, after developing the idea in more generality – this will lead to a discussion of *Bayesian inference*. In the meantime, here is an equivalent problem:

The Prisoner Paradox was posed by Martin Gardner in 1959, and is equivalent to the Monty Hall problem. Here it is, in Gardner's original formulation.[5]

> A wonderfully confusing little problem involving three prisoners and a warden, even more difficult to state unambiguously, is now making the rounds. Three men-A, B and C-were in separate cells under sentence of death when the governor decided to pardon one of them. He wrote their names on three slips of paper, shook the slips in a hat, drew out one of them and telephoned the warden, requesting that the name of the lucky man be kept secret for several days. Rumor of this reached prisoner A. When the warden made his morning rounds, A tried to persuade the warden to tell him who had been pardoned. The warden refused. 'Then tell me,' said A, 'the name of one of the others who will be executed. If B is to be pardoned, give me C's name. If C is to be pardoned, give me B's name. And if I'm to be pardoned, flip a coin to decide whether to name B or C.'
>
> Three prisoners, A, B and C, are in separate cells and sentenced to death. The governor has selected one of them at random to be pardoned. The warden knows which one is pardoned, but is not allowed to tell. Prisoner A begs the warden to let him know the identity of one of the others who is going to be executed. 'If B is to be pardoned, give me C's name. If C is to be pardoned, give me B's name. And if I'm to be pardoned, flip a coin to decide whether to name B or C.'
>
> 'But if you see me flip the coin,' replied the wary warden, 'you'll know that you're the one pardoned. And if you see that I don't flip a coin, you'll know it's either you or the person I don't name.'
>
> 'Then don't tell me now,' said A. 'Tell me tomorrow morning.'
>
> The warden, who knew nothing about probability theory, thought it over that night and decided that if he followed the procedure suggested by A, it would give A no help whatever in estimating his survival chances. So next morning he told A that B was going to be executed.
>
> After the warden left, A smiled to himself at the warden's stupidity. There were now only two equally probable elements in what mathematicians like to call the 'sample space' of the problem. Either C would be pardoned or himself, so by all the laws of conditional probability, his chances of survival had gone up from 1/3 to 1/2.

[5] Gardner, Martin (October 1959). *Mathematical Games: Problems involving questions of probability and ambiguity.* Scientific American. 201 (4): 174–182. Available online: `http://www.nature.com/scientificamerican/journal/v201/n4/pdf/scientificamerican1059-174.pdf`.

The warden did not know that A could communicate with C, in an adjacent cell, by tapping in code on a water pipe. This A proceeded to do, explaining to C exactly what he had said to the warden and what the warden had said to him. C was equally overjoyed with the news because he figured, by the same reasoning used by A, that his own survival chances had also risen to 1/2.

Did the two men reason correctly? If not, how should each calculate his chances of being pardoned? An analysis of this bewildering problem will be given next month.

Exercise 4: Explain how this problem has exactly the same structure as the Monty Hall Problem, and answer Gardner's questions.

6.3 Bayesian Inference

Why, in words, should you switch your guess to Door No. 2? One explanation that you might find compelling is that *Monty had the opportunity to open Door No. 2, and didn't.* So this constitutes some partial evidence that the car is not behind Door No. 2.

Of course, it could be. If the car is behind Door No.1, then Monty *might* have shown you Door No. 3. But if the car was behind Door No. 2, then Monty *definitely* would have shown you Door No. 3. Since Doors 1 and 2 were equally likely to begin with, the fact that Monty showed you Door No. 3 constitutes some partial evidence that the car is behind Door No. 2. It's not a *proof* – but it's enough to make it wise to switch.

This is an example of *Bayesian inference*[6], which consists of the following steps:

1. You are trying to estimate the probability p that some event occurs. You first estimate the *prior probability*, which indicates this probability before any additional evidence is taken into account.

   For example, in the Monty Hall problem, the prior probability is $\frac{1}{3}$ for each of the three doors.

2. You get some additional evidence, which could affect your probability estimate. In the Monty Hall problem, this is the fact that Monty showed you Door No. 3, and it didn't contain a car.

3. You have to figure out the *posterior probability* – a new estimate for the probability which takes into account both your initial estimate and the additional evidence.

   The purpose of this section is to learn how to do this in general.

[6]named after Thomas Bayes (1701-1761), who first put this in quantitatitve form. See *Bayes's theorem* below.

In due course we will introduce *Bayes' theorem.* It is kind of scary looking, and a little bit difficult to remember, but actually it is very simple. It's probably best to make sure you understand the examples, as the theorem is embedded in them.

Example 5: Disease testing.

Suppose you go caving: you explore all sorts of beautiful underground caverns, and have a fabulous time. But afterwards you are alarmed to hear that some cavers catch the (fictional) disease of *cavitosis.* The disease can be treated, and so you decide to be tested to see if you have caught the disease.

Like most medical tests, the cavitosis test is not completely reliable. Instead, it is subject to the following possibilities for error:

- The test has a *false positive* rate of 1%. This means that if you don't have the disease, then with probability 1% the test will (wrongly) say that you do have the disease.
- The test has a *false negative* rate of 3%. This means that if you do have the disease, then with probability 3% the test will say that you don't.

It doesn't *look* so unreliable...

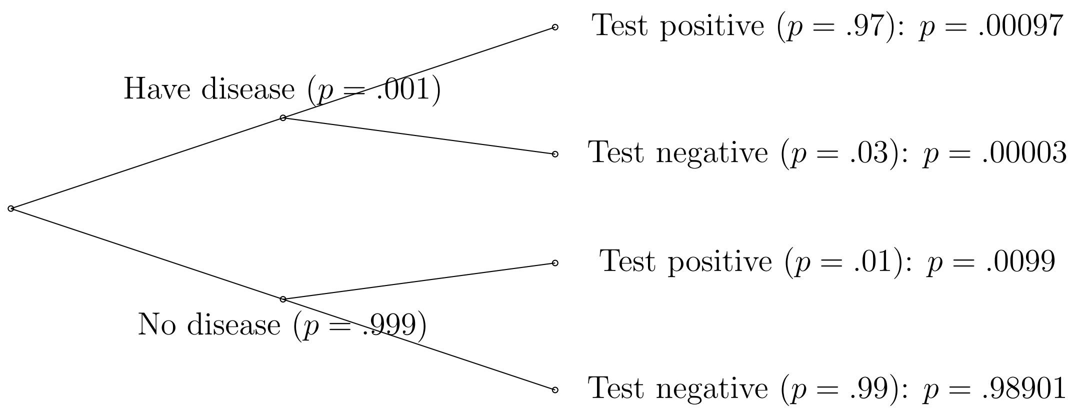
And so our natural question is: *if the test comes back positive, what is the probability that you have the disease?* It is **not** 99% or 97%. Indeed, we can't answer this question yet, because we haven't estimated the prior probability.

Example Continued: Suppose that one in a thousand cavers develop cavitosis. In other words, when you show up to the clinic, you estimate your probability of having cavitosis as $\frac{1}{1000}$.

If the test is positive, what is the probability you have the disease?

Solution. There are four possible events: you have cavitosis and test positive for it; you have cavitosis but test negative for it; you don't have cavitosis but you test positive for it; you don't have cavitosis but you test negative for it.

We compute the probabilities of each of these events in a diagram (a *probability tree*).



Pay close attention to this picture! If you understand it thoroughly, then you can do all the homework exercises.

We first list both of the possibilities for whether you have the disease, with their probabilities. For each of these possibilities, we then list the possible test results, with their probabilities.

Why do we list two probabilities for each of the end results? For example, in the top row, the probability that you test positive is .97, if you have the disease. Since the probability that you have the disease is .001, the probability that you have the disease *and* test positive for it is $.0001 \times .97$.

Note that we have to do the tree in this order! If we tried to draw the tree with the test results first, we would have to know the probabilities of having the disease, *depending on the results of the test.* This is *what we are trying to figure out.*

We're almost done! The probability that you test positive is the sum of the two 'test positive' results in the tree: $.00097 + .00999 = .01096$. Similarly, the probability that you test negative is $.00003 + .98901 = .98904$.

Therefore, by our conditional probability formula, the probability that you have the disease, given that you tested positive for it, is

$$\frac{.00097}{.01096} = 0.0885\cdots = 8.85\%.$$

In other words, you should be concerned, and if your doctor prescribes antibiotics then you should take them, but *it is still more likely than not that you don't have the disease.* In particular, it is much more likely that the test resulted in a *false positive.*

A loose way to think about why this is true? The odds of a false positive are 10 times as large as the odds of actually catching the disease!

Exercise 6: Redo the previous example, assuming this time that (a) one in a hundred cavers develop cavitosis, rather than one in a thousand; (b) one in every *ten* cavers develops cavitosis.

Guess the answers before doing the computations, and compare your results to your guess. (And if they are far off, try to figure out why your guess was wrong.)

Symbolically, we could have expressed the calculation above as follows. Let A be the event that you test positive for the disease, and let B be the event that you actually have it. We want to compute $P(B|A)$, and by the formula for conditional probability we have

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

Moreover, we know that

$$P(A \cap B) = P(A|B)P(B).$$

This is how we computed $P(A \cap B)$, the probability of having the disease and testing positive for it, above: we multiplied the probability of having the disease by the probability that (if you have the disease) you then test positive.

If we substitute the second formula inside the first, we get:

Theorem 7 (Bayes' Theorem): Suppose that A and B are any two events. Then we have

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}.$$

In some sense it is easy to understand: the explanation given just above works for any events A and B. But many people find Bayes's Theorem a bit hard to remember. Perhaps that is because, in typical applications, $P(A)$ is not one of the quantities you're originally given. In the example above, $P(A)$ was the probability of a positive test – taking into account both possibilities for whether or not you have the disease. That is a somewhat strange and unintuitive quantity.

Here is another version of the theorem:

Theorem 8 (Bayes' Theorem – Long Version): Suppose that A and B are any two events. Then we have

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\neg B)P(\neg B)}.$$

Here the symbol $\neg$ means 'not': $\neg B$ is the event that B does *not* occur.

If you stare at this version of the theorem closely, you will see that all of the quantities are given to us, and one can just 'plug everything in'. But it looks a bit intimidating, no? Ultimately the formula is a concise way of explaining how we reasoned with probability trees. Personally, I recommend drawing the trees as opposed to memorizing the formula – it will keep your intuition focused in the right direction.

Here is another example, for which we work out only some of the details. As an exercise, fill in the rest.

Example 9: You speculate that your math teacher may be a brain-devouring zombie. Your prior estimate is that, with 20% probability, he is a zombie.

Zombies also enjoy giving math lectures, and if your teacher is a zombie then with 90% probability he will run class as usual. But with probability 10%, he will instead devour your brains.

Next class, your teacher gives an ordinary lecture and you escape with your brains unscathed. With what probability do you know believe he's a zombie?

Solution. Certainly less than 20%! It goes down, because he could have eaten your brains and he didn't. But even if he were a zombie, he probably would have given an ordinary lecture, so it doesn't go down by much.

Drawing a tree diagram as before (*you should take a minute to do this right now*), there was an 80% prior probability that he is not a zombie, which is the same as the chance that he is not a zombie *and* he didn't eat your brains. There is also a 18% prior probability that he was a zombie but waited until lunch. So the probability of surviving the day was 98%, and the answer is

$$\frac{.18}{.98} = .1836\cdots = 18.36\%.$$

Finally, try this one on your own:

Exercise 10: After the first day, you now believe that there is a 18.36% prior probability that your math teacher is a zombie.

The next class, there is a full moon, which makes zombies especially hungry for brains. You are sick with the flu and don't attend. But your roommate attends the same class and you don't see her again for the rest of the day.

If your math teacher is not a zombie, then you would expect this to happen with probability 5%. Maybe her parents called and she had to go home. But if your math teacher is a zombie, then he definitely devoured your roommate's brains and with 100% probability you would not see your roommate again that day.

With what probability do you now believe your teacher is a zombie?

Elections and polling. You may have paid attention to the recent 2016 U.S. presidential election, between Hillary Clinton and Donald Trump. The election was bitterly contested, and many people who followed it often checked the website **FiveThirtyEight**:

Link: http://fivethirtyeight.com

The website used a mathematical model developed by Nate Silver to maintain a running estimate of the probability of a Clinton or a Trump victory. Whenever pollsters reported the results of opinion polls, the model took this new evidence into account and adjusted the probability appropriately.

Silver's model was extremely complicated, and here we will consider a vastly simplified version. We will also make it a bit more light-hearted: imagine a race for student body president, in which there are two candidates, Amy and Beth. On the eve of their final debate they are tied, and in the debate Amy makes a bold promise: to give every Gamecock a free puppy.

How will students react to this? Maybe they think puppies are adorable and now are eager to vote for Amy Maybe they dread cleaning up all the dog poop, and are thus more inclined to vote for Beth. If you're Nate Silver, you don't try to figure this out – you just pay attention to the polls.

It's important to remember that polls have a *margin of error*. If you poll 1000 people, among a large population which is divided 50-50, then you won't necessarily get exactly 500 of each. This is like flipping 1000 coins: you might get more or fewer than 500 heads, but you'd expect to get *approximately* 500 heads.

We discussed this briefly at the end of the chapter on counting; the *central limit theorem* guarantees that this probability distribution is given approximately by a bell curve. The 'margin of error' describes the range of actual results that could have reasonably accounted for the observed poll results. Often, it is described in terms of a *95% confidence interval*: you can never be *completely* sure what the actual results are, but you can be 95% sure that they are within your margin of error.

This is too complicated to model here, so we will consider the following much simpler version.

Simplified Poll Error Assumption: Assume, of any given poll, that with 60% probability it is *accurate*, with 20% probability it is *three points too high*, and with 20% probability it is *too low*.

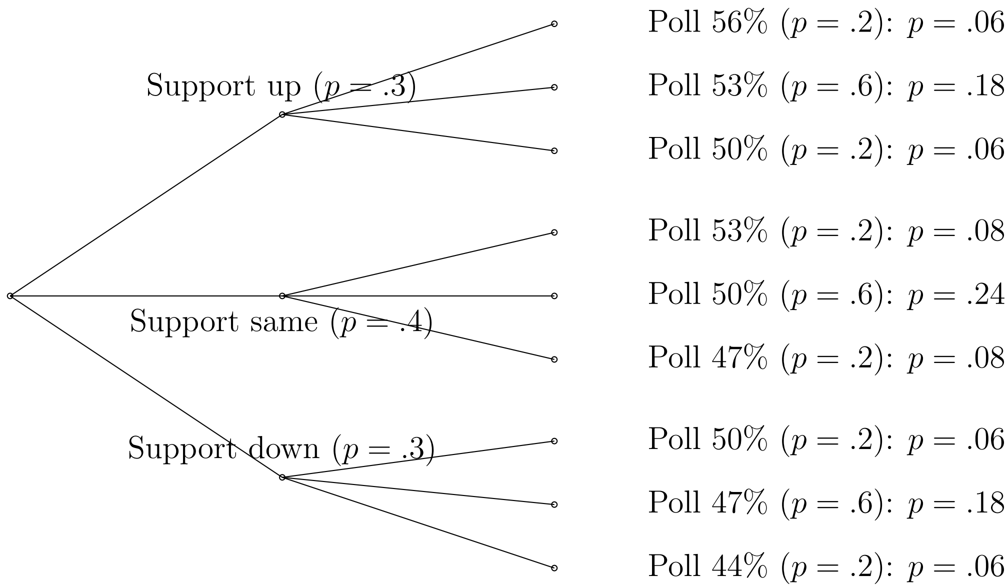
Meanwhile, back to our debate:

Example 11: Estimate, as prior probabilities, that Amy's puppy promise *lowered* her support three points with 30% probability, *raised* it three points with 30% probability, and that with 40% probability it didn't make a difference.

Suppose then that a post-puppy poll comes out showing Amy at 53%. What is the probability that her support actually increased?

Note that you need to somehow estimate these prior probabilities! Otherwise you can't get started.

To begin with, we draw a probability tree again. It is more complicated because it splits into *three* branches at each step.



By Bayes' theorem, we have

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)},$$

where B is the event that her support is at 53%, and A is the event that the poll registers 53% support.

Looking at the table above, $P(A|B)P(B) = P(A \cap B)$ is the probability corresponding to the branch of the tree where both of these events happen – the second branch from the top, with probability .18.

$P(A)$ is the sum of the probabilities of all the branches where she registers 53% support: here there are two, totalling .26. We conclude that the probability of her being at 53% support is now

$$P(B|A) = \frac{.18}{.26} = 69.2\ldots\%.$$

Exercise 12: Compute the probability that her support is still at 50%, and that the poll was too high. (This should be easy, since the tree is already done.)

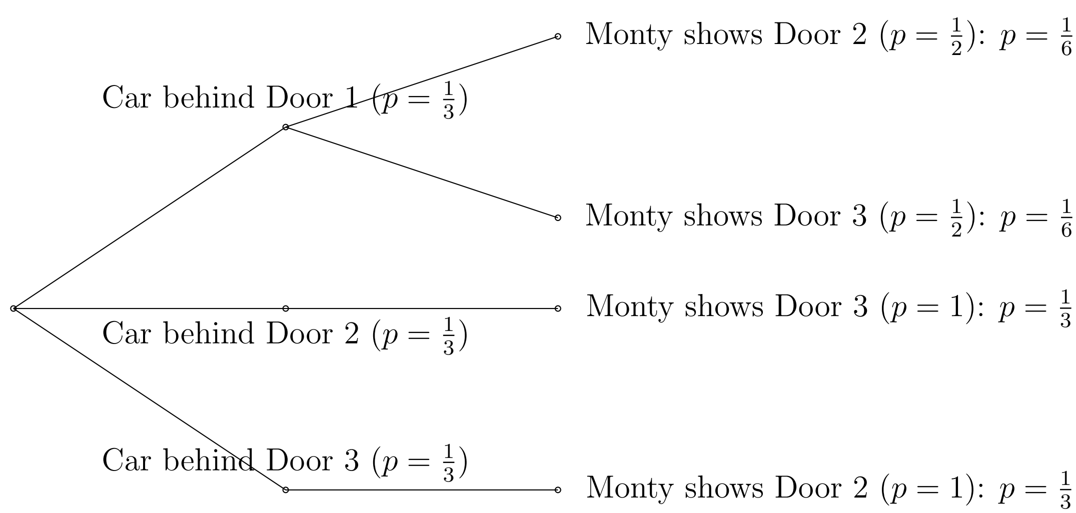
Some further exercises imagine that the campaign drags on, and asks you to further update your estimate of the candidates' support.

This is a *vastly* simplified model. Here we assumed there were only three outcomes, both for the actual result and also for the polling. But in fact there is a whole spectrum of outcomes: Amy's support could have risen 4%, or 2%, or 1.38%, or 0.62%....

A much better model is to use a *continuous* probability distribution function, rather than a discrete one. This means that we have to use calculus. The details are out of the scope of this book, but a course in statistics is highly recommended to anyone who wants to explore this further!

6.4 Monty Hall Revisited

We conclude this chapter with some further discussion of the Monty Hall problem. We begin by giving a solution along the lines of our presentation of Bayes' Theorem.



Note that only the top of the tree has any further branching: it is only if you have chosen the door with the car, that Monty might choose between different options.

As before, suppose that Monty shows you a goat behind Door 3. Again write B for the event that the car is behind Door 2, and A for the event that Monty shows you Door 3; we have

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)} = \frac{1 \times \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3},$$

where as before the $\frac{1}{2}$ in the denominator sums both branches of the tree where Monty showed you Door 2.

This is of course exactly the same thing we did earlier! We just gave a presentation that was more along the lines of our discussion of Bayes's theorem.

The following exercises suggest some variations of the Monty Hall Problem.

Exercise 13: (**Monty Hall with a Million Doors**)

Imagine that the set of Let's Make a Deal has a million doors, rather than 3. One of them contains a car, and all of the rest contain goats.

You choose Door 1, and Monty then opens 999,998 of the remaining doors – all but your door and Door 161,255 – and reveals goats behind them. Explain why you have a winning probability of $\frac{999999}{1000000}$ if you switch.

Do you think you've *finally* understood the Monty Hall problem? Try out these variants, where we make different assumptions about Monty's behavior.

The next two examples are due to Jeffrey Rosenthal[7]. Assume again that there are three doors (not a million). Here we will consider some different assumptions for Monty's behavior:

Exercise 14: (**The Monty Fall Problem**)

In this variant, after you select Door No. 1, Monty slips on a banana peel and *accidentally* pushes open Door No. 3, which just *happens* not to contain the car.

Now the probability that Door No. 1 contains the car has gone up to $\frac{1}{2}$. Why? What is the difference? Explain.

Exercise 15: (**The Monty Crawl Problem**)

You and Monty are standing near Door No. 1, and you choose it. As in the original problem, Monty reveals one of the other two doors which doesn't contain the car.

This time, however, he is very tired and *crawls* from his position to the door he is to open. Since he is lazy, and Door No. 2 is closer, he will always open Door No. 2 if it doesn't contain a car. If Door No. 2 does contain the car, then he will go all the way to Door No. 3.

As before, Monty might open Door No. 2 or Door No. 3. In each case, compute the probabilities that Door No. 1 contains the car.

And, finally, consider these two examples taken from Wikipedia.[8]

Exercise 16: (**The Monty Hell Problem**)

When you appear on the show, you mention that you're a trained mathematician. Monty is feeling mean today and doesn't want to give you a car.

[7] Problem statements taken from his *Monty Hall, Monty Fall, Monty Crawl*, Math Horizons, September 2008. Also available here: `http://probability.ca/jeff/writing/montyfall.pdf`

[8] Taken from the Wikipedia page. See also the very enligthening article by John Tierney in the *New York Times*, July 21, 1991. Available here: `http://www.nytimes.com/1991/07/21/us/behind-monty-hall-s-doors-puzzle-debate-and-answer.html`

So if you pick a door with a goat, then Monty will simply show you the goat immediately, and you lose. But if you pick the door with the car, then Monty will go through his usual shtick and try to get you to switch.

What should you do? What are the probabilities of each of the possible outcomes?

Exercise 17: (Deal or No Deal Variation)

The host of Deal or No Deal guest-hosts an episode. As before you pick one of the three doors. This time, Monty doesn't show you one of the other two doors; instead, he invites you to do so yourself, and if it doesn't reveal the car he gives you the option to switch.

Should you?

6.5 Exercises

The following problems emphasize Bayes' theorem and what it describes about conditional probability. *Instructions*: For each problem, before you work out the details, guess the answer and write down your guess. Then, after you get the answer, compare this to your guess and briefly describe what your competition tells you.

1. Do the exercises in the body of the text – the two zombie teacher exercises, the follow-up exercise to the polling example, and the variations of the Monty Hall problem.

2. Consider the simplified polling example from before. One poll (after Amy's promise of puppies for all) showed Clinton with 53% support, and we computed that this reflects Amy's actual level of support with 69.2% probability, and that the candidates are tied with probability 30.8%.

   Suppose now that a second poll comes in, again showing Amy with 53% support. Now compute the probability that this is her true level of support.

   Finally, suppose that a third poll comes in, this time showing the candidates tied. Now compute the probability that this is her true level of support.

   When FiveThirtyEight adjusts each candidate's probability of winning, this is what it is doing!

3. Recall the experiment we conducted in class: I flipped a coin repeatedly, and it came up heads every time! Eventually it occurred to you that the experiment was rigged – I was ignoring the actual result of the coin flip and just telling you that it was heads every time.

   Assume there is a prior probability of 95% that I was conducting the experiment honestly, and a 5% chance that I was cheating and would always say it comes up heads. For each $n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$, after you have seen n heads, compute the probability that the experiment is rigged.

(Hint: around $n = 4$ or 5, you should definitely suspect it but be unsure. By $n = 10$, there is a small possibility I am telling the truth, but at this point you should be reasonably confident that I'm lying.)

4. You decide to conduct your own experiment: You pull a coin out of your pocket, keep flipping it, and it keeps coming up heads over and over!

   This time, since you produced the coin yourself, you estimate the prior probability as one in a million that by some freak chance this coin had heads on both sides. Now estimate the probability of this, after $n = 0, 5, 10, 15, 20, 25, 30$ flips.

5. Here is a clip of the game Barker's Markers (from The Price Is Right):

   Link: Barker's Markers – The Price Is Right

   (a) Assume that the contestant has no idea how much the items cost and guesses randomly. Also assume that the producers choose randomly two of his correct guesses and reveal them.

   Explain why he has a winning probability of $\frac{3}{4}$ if he switches. (Use Bayes' theorem!)

   (b) Watch several clips of this game (also known as 'Make Your Mark' during the Drew Carey era). Determine, as best as you can, the extent to which these assumptions are accurate. In the clips you watch, do you think the contestant should indeed switch?

6. Compose, and solve, your own problem that involves Bayes's theorem, similar to the above. (Bonus points for problems which are particularly interesting, especially realistic, or are drawn from any game show.)

7. **Term project.** Send me, **by e-mail**, a rough indication of what you would like to do for your term project. You are free to ask me questions, or to suggest more than one idea if you would like advice on which is the most feasible.

   Unless you prefer to work alone, please let me know whom you plan to work with (if you know). If you don't know, I'm happy to match people who have indicated similar interests.

7 Competition

7.1 Strategic Games

The following clip is from the final round of the British Game Show *Golden Balls*.

Link: Golden Balls

Game Description (Golden Balls – Final Round): Two players play for a fixed jackpot, the amount of which was determined in earlier rounds. They each have two balls, labeled 'split' and 'steal'. They are given some time to discuss their strategies with each other. Then, they each secretly choose one of the balls and their choices are revealed to each other.

If both choose the 'split' ball, they split the jackpot. If one chooses the 'split' ball, and the other 'steal', the player choosing 'steal' gets the entire jackpot. If both players choose 'steal', they walk away with nothing.

In the video, the players are competing for a prize of 8,200 pounds (roughly $11,000 in US currency). We can summarize the game in a two-by-two grid that describes the possible choices for you and for your opponent, and the outcome of each choice.

		You	
		Share	Steal
Opponent	Share	5500, 5500	11000, 0
	Steal	0, 11000	0

The outcomes are listed for you and your opponent respectively.

Now, *let us assume that you only to maximize your own expected payoff.* This is a big, and not always realistic, assumption: it assumes that you don't care about your opponent one way or another, and that you don't care if you look like a jerk on national television.

If you assume this, then your optimal strategy is clear. If your opponent steals, it doesn't matter whether you steal or share. If your opponent splits, then you do better if you steal. *Therefore, it is clear that you should always choose the steal ball.*

Of course, your opponent will reason in exactly the same way. Therefore, your opponent will deduce that she should choose the steal ball, *and therefore with optimal strategy both of you will choose 'steal' and win nothing.*

Here is an amusing video of the same game being played by two other contestants:

Link: Golden Balls – Promise to Steal

You can also find other videos of this game on YouTube, many of which will do somewhat less to affirm your faith in humanity.

Setup and notation. This is an example of a two-player *strategic game*, of the type studied in the subject known as *game theory*. Although this entire course is about games, mathematical 'game theory' refers to this sort of analysis: you have a game involving two or more players, and you have distilled the analysis down to the point where you know what will happen depending on your, and your opponents', choice of strategy.

Definition 1 (Strategic Games): A *strategic game* consists of the following inputs:

- Two or more players. Here we will only look at two player games.
- For each player, two or more choices of strategy. Each player chooses their strategy independently and simultaneously.

  For example, in the Golden Balls example, each can choose 'Share' or 'Steal'. This particular game happens to be *symmetric*: the strategies and payoffs are the same for each player.
- A *payoff matrix*, such as the one above, describing the payoff to each player. In this book, the *columns* will correspond to the *first* player's strategies, and the *rows* to the *second*.

Golden Balls is an example of a *non-zero-sum game*:

Definition 2 (Zero-Sum Games): A strategic game is called *zero-sum* if the payoffs for all the players always add to zero, no matter what strategies they choose.

By abuse of language, we will also call a game zero-sum if the payoffs for all players add to some fixed amount, independent of their choice of strategy.

Most familiar games are zero-sum: if the Gamecocks win a football game, then their opponents lose, and vice versa. Board games are usually also zero-sum: there is one winner, and everyone else loses. But there are exceptions: for example, *Pandemic* is a cooperative game in which everyone either wins or loses together. *Republic of Rome* is another, which allows for at most one winner – but if the players don't cooperate to some extent, then Rome falls and everyone loses.

Golden Balls is a typical example of a *non-zero-sum* game: they are playing for a total of £8,200, but both players could lose.

This chapter is about finding optimal strategies for strategic games. What should 'optimal' mean? Here is one possible definition:

Definition 3 (Dominant Strategy): A strategic game has a *dominant strategy* if that strategy is always at least as good as any other strategy, no matter how your opponent plays.

If you are trying to maximize your payoff, then you should clearly play a dominant strategy – if there is one. As we will see, there often isn't.

7.2 Examples of Strategic Games

We now give some common examples of strategic games.

Example 4 (The Prisoner's Dilemma): You are a member of a criminal gang. You and a colleague are caught, arrested, and imprisoned in separate cells – with no way of communicating with each other.

Right now, prosecutors lack sufficient evidence to convict either of you on a big charge, but they have enough evidence to convict you on a lesser charge. They offer you and your colleague a bargain: you are each given the opportunity to betray the other, by testifying that the other committed the crime. Suppose you know that:

- If you each betray the other, you each serve 2 years in prison.
- If you betray your colleague, and (s)he remains silent, you will be set free and your colleague will get 3 years.
- If you both remain silent, you both will serve 1 year in prison (on the lesser charge).

What should you do?

The payoff matrix for this game as follows:

		You: Betray	You: Silent
Colleague	Betray	-2, -2	-3, 0
	Silent	0, -3	-1, -1

Mathematically speaking, the game is easy to analyze. For each prisoner, betrayal is a dominant strategy, and each prisoner should betray the other.

This invites a very serious paradox, as well as questions about whether the model is realistic. (If you betray your accomplice and (s)he remains silent, you had better get out of town as soon as you are released from jail.) It is also interesting if the game is played multiple times consecutively. But we won't pursue these questions here.

Example 5 (Rock, Paper, Scissors): This is a familiar game and we can describe its payoff matrix immediately.

		A: Rock	A: Paper	A: Scissors
B	Rock	0, 0	1, -1	-1, 1
B	Paper	-1, 1	0, 0	1, -1
B	Scissors	1, -1	-1, 1	0, 0

This is a stupid game of pure luck, but don't tell these people:

Link: Rock-Paper-Scissors Tournament

Note that this is also an example of a *zero-sum game*: a win for you is equivalent to a loss for your opponent.

There is no dominant strategy, which is best against all opponents' strategies simultaneously.

The game of **Chicken** is illustrated by this clip[9] from the movie *Rebel Without a Cause*.

Link: Rebel Without a Cause

Example 6 (Chicken):

Two teenagers challenge each other, in front of all of their friends, to the following contest. They start far apart and drive their cars at maximum speed towards each other. If one swerves and the other one does not, then the driver who swerves is the 'chicken' and loses face among his friends, while the other enjoys increased social status.

If neither swerves, the resulting accident kills them both.

The payoff matrix for this game might be described as follows:

		A: Swerve	A: Straight
B	Swerve	0, 0	1, -1
B	Straight	-1, 1	-100, -100

[9]Thanks to Kevin Bartkovich, who taught me this subject and who used this very clip for this example.

Once again, there is no dominant strategy.

Here is a clip of the final round from the game show **Jeopardy**:

Link: Final Round – Jeopardy

> **Game Description (Jeopardy – Final Jeopardy):** Three players come into the final round with various amounts of money. They are shown a category and write down a dollar amount (anything up to their total) that they wish to wager.
>
> After they record their wagers, they are asked a trivia question. They gain or lose the amount of their wager, depending on whether their answer was correct. Only the top finisher gets to keep their money.

Since the game is quite complicated to analyze in full, we consider the following toy model of it:

> **Example 7 (Jeopardy – Toy Model):** Two players, A and B, start with 3 and 2 coins respectively. Each chooses to wager none, some, or all of her coins, and then flips a coin. If the coin comes up heads, she gains her wager; tails, she loses it.
>
> How should each player play to maximize her chances of winning (counting a tie as half a win)?

We can compute the payoff matrix for the game as follows:

		A: 3	A: 2	A: 1	A: 0
B	2	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{5}{8}$	$\frac{4}{8}$
B	1	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{6}{8}$
B	0	$\frac{4}{8}$	$\frac{4}{8}$	$\frac{6}{8}$	1

As the question was asked, this is a zero-sum game: a win counts as 1, and a loss counts as 0. For brevity's sake, we only list the payoffs for A; the payoffs for B are one minus these.

The table takes a little bit of work to compute. For each combination of wagers there are four possibilities: A and B both gain, A and B both lose, only A gains, only B gains. We add $\frac{1}{4}$ for each such scenario in which A wins, and $\frac{1}{8}$ for each such scenario in which they are tied.

This is our first example of an *asymmetric* game. Careful inspection of the table reveals that there is no dominant strategy. There is, however, a *dominated* strategy, namely a wager of 3 for A. A always does as well or better by wagering only 2 instead. So we can at least eliminate the 3 column from the table.

Between the other strategies it is not obvious what the players should choose: your best strategy depends on your opponent's best strategy, in a somewhat complicated way. (The same is true on the actual game show.) We will come back to this later.

Exercise 8: Rather than computing each player's chances of winning, compute the payoff matrix using the expected dollar value of each player's winnings. How does this game differ from the previous one? How is it similar? Is this a zero-sum game?

7.3 Nash Equilibrium – Pure Strategies

In this section, we will begin to further analyze the question of finding an 'optimal strategy' for a game. But before we proceed, we remind the reader of our standing assumptions: that the game will be played *once only*, and that each player wants to maximize their own payoff.

It is very interesting to consider how game theory changes when games are played repeatedly, or when additional factors (such as altruism or spite) are involved. Occasionally, we will consider such questions. But for the most part they are outside the scope of this book.

We begin with the following easy considerations:

- If both players have a dominant strategy, then they should both play it.
- If only one player has a dominant strategy, then she should play it. The other player should choose whichever strategy does best against this dominant strategy.

But what if neither player has a dominant strategy? For example, consider again the payoff matrix for Chicken:

		A: Swerve	A: Straight
B	Swerve	0, 0	1, -1
B	Straight	-1, 1	-100, -100

If your opponent swerves, you should go straight. And if your opponent goes straight, you should swerve. So there is no 'optimal strategy'. But there are *Nash equilibria*:

Definition 9 (Nash Equilibrium – Pure Strategies): Suppose you and your opponent each choose a (pure) strategy for a game. Then these strategies form a (pure strategy) **Nash equilibrum**[10] if:

- your current strategy is optimal against her current strategy, and
- her current strategy is optimal against your current strategy.

In other words, neither of you can do better by individually switching your strategy. Or, to say the same thing yet another way, given your strategy, your opponent can't do any better than her current strategy, and vice versa.

Remark: In the next section we will discuss *mixed*, or randomized, strategies. A *pure* strategy is simply what we have been calling a 'strategy' above.

For example, in Rock, Paper, Scissors, the pure strategies are 'rock', 'paper', and 'scissors'; in Chicken, the pure strategies are 'swerve' and 'straight'. An example of a mixed strategy might be to play 'rock', 'paper', or 'scissors' with probability $\frac{1}{3}$ each.

Hopefully the following is clear:

Theorem 10 (Dominant Strategies are Nash Equilibria): If both players have dominant strategies, then these strategies form a Nash equilibrium.

In other words, dominant strategies always give Nash equilibria, but not vice versa. This is nicely illustrated by our game of Chicken.

Example 11: Find all the pure strategy Nash equilibria for the game of Chicken.

Solution. Suppose you choose to swerve. To optimize against your strategy, your opponent should choose to go straight. Since your choice (swerve) is also optimal against her choice (straight), these strategies form a Nash equilbrium.

If you choose to go straight, then your opponent should swerve, and these choices also form a Nash equilbrium.

This is all; there are two pure strategy Nash equilbria. There is also a *mixed strategy* Nash equilibrium (see the exercises).

Example 12 (Stag Hunt): You and another player have to choose to either cooperate and hunt a stag, or to hunt a rabbit on your own. You can catch a rabbit by yourself, but you need the other player's cooperation to successfully hunt the stag. The payoff matrix is as follows.

		You	
		Stag	Rabbit
They	Stag	2, 2	1, 0
	Rabbit	0, 1	1, 1

What are the Nash equilibria?

"Obviously", you should both should cooperate to hunt the stag: this results in the best possible payoff to both players. This is a Nash equilbrium: if one player has decided to hunt the stag, it's in the other's best interests to cooperate.

However, this is not a dominant strategy: if for whatever reason the other player chose to hunt the rabbit, then you should do the same. So both players hunting the rabbit is also a Nash equilibrium.

The following illustrates an example of a *coordination game*.

Example 13 (A Coordination Game): You and another player place a coin heads or tails, each invisibly to the other. You have no chance to communicate beforehand. If you both make the same choice, then you get a reward; otherwise, nothing happens.

The payoff matrix for this game is as follows.

		You	
		Heads	Tails
They	Heads	1, 1	0, 0
	Tails	0, 0	1, 1

The game has two Nash equilibria: you both pick heads, or you both pick tails. This is somewhat similar to the Stag Hunt, but here it is not obvious which you should pick.

Here is another coordination game, with a twist.

Example 14 (Battle of the Sexes): A husband and wife agreed to meet on a Saturday, either at a football game or the opera. But they forgot where they agreed to meet, and they have no chance to communicate beforehand. So each chooses, independently, to go to the football game or the opera.

Each partner's first priority is to be with the other. But, given that, the husband would rather attend the football game, and the wife would rather attend the opera. A payoff matrix for the game is as follows:

		Husband	
		Football	Opera
Wife	Football	3, 2	0, 0
	Opera	0, 0	2, 3

There are two Nash equilibria: both attend the football game, or both attend the opera.

The game is a bit contrived (*do they not have cell phones?*), and arguably sexist (some men like opera, and some women like football), but memorable nevertheless.

Not every game has a pure strategy Nash equilbrium:

Example 15 (Rock, Paper, Scissors): The game Rock, Paper, Scissors has *no* pure strategy Nash equilibria.

Why is this? Suppose, for example, that you choose Rock. Then, to optimize against your strategy, your opponent will choose Paper.

But then your strategy isn't optimal against hers! If she chooses Paper, then you should switch your strategy to Scissors. But then she should switch to Rock. And then you should switch to Paper. And so on, forever.

Another way to see this: a Nash equilbrium requires that your strategy be optimal against your opponent's current strategy, *and* that her strategy be optimal against your current strategy. But in Rock, Paper, Scissors, these can't be true. If your strategy is optimal against your opponent's current strategy, then hers is *worst* possible against yours.

As we will see in the next section, this game does however have a *mixed strategy* Nash equilibrium: each player should randomize, and choose Rock, Paper, and Scissors each with probability $\frac{1}{3}$.

7.4 Nash Equilibria - Mixed Strategies

We now describe how to find Nash equilibria involving mixed strategies.

The idea behind mixed strategies to include a random element into your decision making, to make you more difficult to predict (and therefore exploit). For example:

- In Rock, Paper, Scissors, you may play each strategy with probability $\frac{1}{3}$. Then you cannot be exploited, no matter how clever your opponent is.
- Imagine a game of poker, where you are considering raising or folding. If your hand is strong, then you should raise.

  If your hand is weak, then it may be sensible to fold. But you might consider bluffing: if your opponent's hand is weak or medium-strength then *he* might fold to a raise.

  The best strategy is a mixed strategy: you should fold sometimes and raise sometimes. (Indeed, in his *Harrington on Hold 'em*, Dan Harrington recommends wearing a watch to poker games, so that you can use the second hand as a random number generator.) If you play unpredictably, then your opponent will be unable to deduce your hand from your choices, and will thus be led to make mistakes.

Formally by a *mixed strategy* we mean the following:

Definition 16 (Mixed strategy): By a **mixed strategy** we mean an assignment of a probability (between 0 and 1, inclusive) to each possible strategy.

Generally, you should play a mixed strategy if you want to ensure that you cannot be exploited. Many games have Nash equilibria where both players choose a mixed strategy! In this section we will explain how to find these Nash equilibria.

We first have to generalize the notion of a Nash equilibrium to allow for mixed strategies:

Definition 17 (Nash Equilibrium, including mixed strategies): Suppose you and your opponent each choose a (pure or mixed) strategy for a game. Then these strategies form a **Nash equilibrum**[11] if:

- your current strategy is optimal against her current strategy, and
- her current strategy is optimal against your current strategy.

Remark: Remember that 'optimal' allows for ties. We don't require that an optimal strategy be *better* than every other strategy; we only require that it be *at least as good as* every other strategy.

In this generality, there is *always* a Nash equilibrium:

Theorem 18 (Existence of Nash Equilibria): Every game has at least one (pure or mixed strategy) Nash equilibrium.

This is true no matter how many players are involved, and no matter how many strategies the players have available![12] We will (mostly) focus on the case of two players, where each player has two strategies available.

Finding Nash Equilibria – Two Players, Two Strategies: Suppose you are analyzing a game with two players, and two strategies available to each. Then you can find all the Nash equilibria as follows:

- First, find the *pure strategy* equilbria. One way to do this: look at each strategy for Player 1 in turn, and see if it is in Nash equilibrium with some strategy for Player 2.
- If neither player has a dominant strategy, also look for a Nash equilbrium where each player chooses a mixed strategy. In this case, each player will choose their strategy so as to leave their opponent indifferent among all possible choices of strategy.

The recipe for finding mixed strategy Nash equilibria might look a bit strange. The following example will explain it.

Example 19: Consider again the payoff matrix for *Battle of the Sexes.*

		Husband: Football	Husband: Opera
Wife	Football	3, 2	0, 0
Wife	Opera	0, 0	2, 3

This game has three Nash equilibria. Two of them involve pure strategies: both go to the football game, or both go to the opera.

There is also a mixed strategy Nash equilibrum: each player chooses their preferred activity with probability $\frac{3}{5}$. We will explain how to compute this shortly; for now we begin by explaining why it is a Nash equilbrium. We need to justify two claims:

[12] We do, however, require that there be *finitely many* players and strategies.

- Given that the husband chooses to go to the football game with probability $\frac{3}{5}$, an optimal strategy for the wife is to go to the opera with probability $\frac{3}{5}$.
- Conversely, given that the husband chooses to go to the opera with probability $\frac{3}{5}$, an optimal strategy for the wife is to go to the football game with probability $\frac{3}{5}$.

We say 'an optimal strategy' rather than 'the optimal strategy' because each strategy will be equally good! Consider the first bullet. If the husband chooses to go to the football game with probability $\frac{3}{5}$, and the wife attends the football game, then her expected payoff is

$$\frac{3}{5} \cdot 2 + \frac{2}{5} \cdot 0 = \frac{6}{5}.$$

If alternatively she attends the opera, then her expected payoff is

$$\frac{3}{5} \cdot 0 + \frac{2}{5} \cdot 3 = \frac{6}{5}.$$

So her payoff is the same for either of the pure strategies! It is also the same for *any* mixed strategy. Therefore, *given the husband's choice of mixed strategy*, any choice for the wife is equally good.

An analogous computation shows that, given the wife's current mixed strategy, any choice is equally good for the husband. That means that neither player has an incentive to change their strategy, so the game is in Nash equilibrum.

Remark: Being at Nash equilibrium doesn't mean that you can't *both* do better by *both* adjusting your strategy. For example, in Battle of the Sexes, either of the pure strategy equilibria is better for *both* players than the mixed strategy equilibrium described above.

Games can have more than one Nash equilibrium, and one equilibrium may be better for one or both players than another.

Nash equilibrium addresses only the prospect of one player changing their strategy at a time. As we will see, this makes things more practical to analyze.

Example 20: In Rock, Paper, Scissors, the unique Nash equilibrium is a mixed strategy for both players: both players choosing rock, paper, and scissors with probability $\frac{1}{3}$ each. From each player's standpoint, the game is a draw (on average) no matter what, and so neither player has any incentive to switch.

There is no other Nash equilibrium for this game. Why? Suppose for example that Player A chooses any other strategy. Then it will be biased in favor of one of the three possible moves – let's say Rock. Then your opponent should switch strategy and exploit you by always playing Paper.

We now describe how to find all the Nash equilibria for two-player, two-strategy games. Consider the following game:

Example 21 (Less Morbid Chicken): This game is modeled after Chicken, but without death as a possible outcome.

Two players have the opportunity to be *nice* or *greedy*. If both players are nice, each gets a dollar. If only one is greedy, that player gets both dollars. But if both players are mean, they each pay a penalty of five dollars.

The game is described by the following payoff matrix.

		A: Nice	A: Greedy
B	Nice	1, 1	2, 0
B	Greedy	0, 2	-5, -5

We begin by finding the pure strategy Nash equilibria. Both playing nice is *not* an equilibrium: you should defect, and take your partner's money! Similarly, both playing greedy is also not an equilibrium: you should give in and play nice, so as to avoid the bottom right corner.

However, you may check that it *is* a Nash equilibrium for you to always be nice and your opponent to always be greedy, or vice versa. If your opponent is always greedy, then you stave off the damage by always being nice. Conversely, if you decide to always play nice, then – in the dog-eat-dog world of theoretical game theory, there is no reason for your opponent not to exploit you.

There is also a mixed strategy Nash equilbrium. To find it, we introduce a parameter α: suppose your opponent plays a mixed strategy where she is nice with probability α, and greedy with probability $1 - \alpha$. If this is half of a Nash equilibrium, then the payoffs to you for being greedy and for being nice will be the same.

Given your opponent's strategy, the expected value of you being nice is

$$1 \cdot \alpha + 0 \cdot (1 - \alpha) = \alpha,$$

and the expected value of you being greedy is

$$2 \cdot \alpha + (-5) \cdot (1 - \alpha) = 7\alpha - 5.$$

If $\alpha > 7\alpha - 5$, then you should always be nice; if $\alpha < 7\alpha - 5$, then you should always be greedy.

The interesting case is when these are equal, i.e., when $\alpha = 7\alpha - 5$. Doing the algebra, this is the same as saying $6\alpha = 5$, or $\alpha = \frac{5}{6}$. If your opponent is nice with probability $\frac{5}{6}$, then every strategy has the same payoff: the extra dollar you get from being mean exactly balances the occasional big loss.

Conversely, since the game is symmetric (i.e. it looks the same from the standpoint of both players), you choosing to be nice with probability $\frac{5}{6}$ will leave your opponent indifferent to changing her strategy.

So, *both players being nice with probability $\frac{5}{6}$, and greedy with probability $\frac{1}{6}$, forms a mixed strategy Nash equilbrium.*

We can compute the expected payoff to both players of these strategies. The probabilities of each combination are

		A: Nice	A: Greedy
B	Nice	$\frac{25}{36}$	$\frac{5}{36}$
	Greedy	$\frac{5}{36}$	$\frac{1}{36}$

and so we can compute the expected payoff of the game using a standard expected value computation. The expected payoff is

$$\frac{25}{36} \cdot 1 + \frac{5}{36} \cdot 2 + \frac{5}{36} \cdot 0 + \frac{1}{36} \cdot (-5) = \frac{30}{36} = \frac{5}{6}$$

to each player. It is not as good as if you both play 'Always Nice', but with this choice of strategy you do pretty well, *and* you ensure that you cannot be exploited.

7.5 A More Sophisticated Example

We close this section by analyzing our model of Final Jeopardy. This involves one more concept, namely that of *iterated elimination of dominated strategies.* There are some important technicalities which are discussed in more comprehensive accounts of game theory — here we opt for a treatment which is as simple as possible, and find our way to a Nash equilbrium.

Recall our game: A has 3 coins, and B starts with two. Each places a wager and either wins or loses that many coins with 50-50 probability. Each player's objective is to finish with more coins than their opponent; the actual number of coins they finish with is not important. (A tie counts as half a win.)

As we computed earlier, the payoff matrix describes Player A's chances of winning. For simplicity, we give only the payoffs to Player A; to compute the payoffs to Player B, subtract these numbers from 1.

		A		
		2	1	0
	2	$\frac{6}{8}$	$\frac{5}{8}$	$\frac{4}{8}$
B	1	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{6}{8}$
	0	$\frac{4}{8}$	$\frac{6}{8}$	1

Recall that we didn't include the possibility that A wager all three coins, because it was *dominated* by the choice of wagering two coins. Wagering two is at least as good in all situations, and in some situations better. So we can *eliminate* this strategy.

We now introduce an essentially cosmetic change: we multiply all the payoffs by 8 and subtract 4. This results in a nicer-looking payoff matrix, without changing the essence of the problem.

		A		
		2	1	0
	2	2	1	0
B	1	1	2	2
	0	0	2	4

Our aim is to find a Nash equilibrium. You can quickly see that there are no pure strategy Nash equilibria. (This is a *zero-sum game*, so you shouldn't expect one.) So we will seek a mixed strategy equilibrium.

We can now eliminate 1 *as a strategy for B*. The reason for this is that *it is dominated by a mixed strategy of playing* 0 *or* 2 *with probability* $\frac{1}{2}$ *each.* If B plays 1, then the payoffs *to A* are respectively 1, 2, and 2 depending on A's strategy. But if B plays 0 or 2 with probability $\frac{1}{2}$ each, then the average payoffs to A are respectively 1, 1.5, and 2: at least as good in all cases, and better in one case.

So, in other words, if B is thinking of playing 1, then she should instead flip a coin and play 0 or 2. The elimination of this strategy simplifies the payoff matrix:

		A		
		2	1	0
B	2	2	1	0
	0	0	2	4

We can now go one step further by eliminating 1 as a strategy for A also. In this case, it is *exactly equivalent* to wagering 0 or 2 with 50-50 probability. Eliminating this strategy from A's choices, we obtain:

		A: 2	A: 0
B	2	2	0
	0	0	4

Finally, we can find the Nash equilibrium! We want a mixed strategy. So suppose B plays 2 with probability p and 0 with probability $1-p$. Then the payoff to A from playing 2 is

$$2p + 0(1-p) = 2p,$$

and the payoff to A from playing 0 is

$$0p + 4(1-p) = 4(1-p).$$

In a mixed strategy Nash equilibrium, A will be indifferent to these two strategies, so we must have $2p = 4 - 4p$ – solving this yields $p = 2/3$. So B should play 2 with probability $\frac{2}{3}$ and 0 with probability $\frac{1}{3}$.

Conversely, suppose A plays 2 with probability q and 0 with probability $1-q$. Then the payoff to A (i.e. the negative payoff to B) from playing 2 is

$$2q + 0(1-q) = 2q,$$

and the payoff to A from playing 0 is

$$0q + 4(1-q) = 4(1-q).$$

So, similarly, A should play 2 with probability $\frac{2}{3}$ and 0 with probability $\frac{1}{3}$.

7.6 Exercises

1. Consider our original game of Chicken:

		A: Swerve	A: Straight
B	Swerve	0, 0	1, -1
	Straight	-1, 1	-100, -100

Compute the Nash equilibrium for this game (it will be a mixed strategy for both players) and the expected payoff.

2. Consider a variation of Battle of the Sexes where the husband *really* likes football. Suppose that it has the following payoff matrix:

		Husband: Football	Husband: Opera
Wife	Football	10, 2	0, 0
	Opera	0, 0	2, 3

   Note that the game is no longer symmetric.

   Determine all the Nash equilibria of the game, and the expected payoff of each.

3. Consider a different variation of Battle of the Sexes, where both partners prefer the opera to football.

		Husband: Football	Husband: Opera
Wife	Football	2, 2	0, 0
	Opera	0, 0	3, 3

   Again determine all the Nash equilibria and the expected payoffs. Your answer should be very different from the previous one! Why?

4. In a third version, suppose that the husband is a jerk, and that he just wants to watch the football game and doesn't care about his wife. Now the payoff matrix is the following:

		Husband: Football	Husband: Opera
Wife	Football	3, 2	0, 0
	Opera	3, 0	0, 3

   Again determine all the Nash equilibria and the expected payoffs. Describe your conclusions.

8 Backwards Induction

The most famous example of 'backwards induction' might be the following:

Example 1 (Ice Cream Division): You (as a child) and your sister are to split a small amount of ice cream. How can you split it fairly?

Here's the classical solution: you split it into two parts, and allow your sister to choose first. The optimal strategy is to split it into two *equal* parts: if one was larger than the other, then your sister would take that one.

Why is this fair? To analyze the game we use *backwards induction* – a process we have seen earlier (for example in our analysis of Punch-a-Bunch). This means that we have to reason, starting from the end of the game, and working backwards.

In this case it is easy: since your sister will choose the larger bowl, you want to make the *smallest* bowl as large as possible. So you prepare two bowls of equal size.

Here's another example:

Example 2 (Nim): The game of *Nim* is played with any number of pennies. Two players take turns, and on each turn a player must take exactly one, two, or three pennies. The player forced to take the last penny loses.

Which player has a winning strategy? What is it?

If you start with, say, a hundred pennies, then it is not obvious what to do! But, in the endgame, there will be only a few pennies left. This game becomes easier to analyze when we start at the end and work backwards:

- (2, 3, 4) If there are 2, 3, or 4 pennies left over, and it's your turn, then you win by taking all but one. Your opponent will be forced to take the last penny.
- (5) If it's your turn with exactly 5 pennies left over, then you can't win unless your opponent makes a mistake: depending on your move, you will leave your opponent with 2, 3, or 4 pennies – and we saw that these are all winning positions for your opponent.
- (6, 7, 8) If it's your turn with 6, 7, or 8 pennies, then you win by taking all but five. Your opponent will then be left with a losing position.
- (9) If it's your turn with 9 pennies, then you lose: once again, you are forced to leave your opponent with a winning position.
- (10, 11, 12) You win by leaving nine pennies.
- And so on....

So the losing positions are $1, 5, 9, 13$, and so on: the numbers which are one plus a multiple of 4. Anything else is a winning position.

Example 3: You share a prison cell with a fellow prisoner. You know each other to both be very intelligent, and well versed in backwards induction.

One day the jailer comes and paints a mark on each of your foreheads – either red or blue. You don't have any idea what color your mark is, but you can see your cellmate's – it is red. He can also see yours.

The jailor informs you both that either (or both) of you may guess the color of your mark. If you guess right, you will be set free, but if you guess wrong, you will be executed. You would very much like to be set free, but you even more don't want to be executed, so neither of you is willing to guess unless you are certain.

Finally, the jailor then tells you: 'At least one of you has a red mark'. After a few moments, you both raise your hand and each inform the jailor – correctly – that your forehead has a red mark. You both go free. *How did you know?*

The solution is to consider the problem from your cellmate's perspective. Suppose instead that you had a blue mark. Then your cellmate would see your blue mark. Since he knew that *at least one of you had a red mark*, and *it's not you*, it therefore must be *him*. So he would have *immediately* guessed that his own mark was red.

He did not do so; therefore your mark is not blue. So you can guess with confidence that it is red.

In this chapter we will consider a variety of examples that can be analyzed by such techniques.

8.1 Division Games

We now consider a class of games generalizing the 'ice cream game' presented at the beginning, and for which the optimal strategy can be found by backwards induction.

Example 4: Two or more players play to divide a pot of \$100.

The first player proposes any division of the money among all the players. Then, in turn, each player can either (a) *accept* the previous player's proposed division, on behalf of all the players, or (b) *refuse* it and propose a different one.

The last player doesn't have any opportunity to make an offer; instead, if it gets to her and she refuses the proposed offer, then all the money is divided equally.

Example. In a 3-player game, player 1 proposes that the money be divided 34/33/33. Player 2 refuses the offer, and proposes a division of 0/50/50. Player 3 accepts: the last two players split the \$100 evenly, and the first player gets nothing.

What is the optimal strategy for each player, assuming that each plays rationally to maximize their own self-interest? Consider first the 3-player game. We work backwards:

- The third player can always win 33.33 by splitting the pot evenly. She will accept the second player's offer if and only if it gives her more than 33.33.
- The second player can offer 0/66.66/33.34, according to our assumptions the third player will accept this. (In practice, she might be annoyed at the second player's greed, and give up the extra penny and split the money equally.)

  The second player will do this, unless she is presented an offer guaranteeing her more than 66.67.
- The first player knows that she will get nothing, unless she offers the second player at least 66.67. Therefore she does so, and offers 33.33/66.67/0. Subject to our assumptions, the second player will accept – this is the outcome of our game.

Remark: Now let's change our assumptions, and say that the third player doesn't play strictly to maximize her own payoff: she will not accept any offer in which someone else is better off than her. So, in particular, she would reject the 0/66.66/33.34 offer described above and instead choose to split the money equally.

We still assume that the first two players act rationally, and they're familiar with the third player's preferences. Then this changes everything! The second player will offer 0/50/50, which means that the first player should offer 50.01/49.99/0. So, in this example, the third player still gets nothing, but the first two players split the prize in different proportions.

Exercise 5: Returning to our standard assumption, what will happen in a 4-player game?

Example 6 (Pie Division Game): Two or more players divide a pie. The first player begins by taking some or all of the pie.

The second player then *either* takes some or all of the remaining pie, *or* takes some or all of *the first player's* pie.

Similarly, each remaining player in turn chooses some or all of the remaining pie, or some or all of any other player's pie.

Each player's objective is to have as much pie at the end as possible.

We will not give a detailed analysis of this game here, instead leaving it to the reader. You may check that, if all players play optimally, then the following will happen: each player will, in turn, take a tiny sliver less than their fair share of the pie. This will make sharing the remaining pie the most tempting option for the remaining players. So, with optimal strategy, no stealing will occur, and all players will get a nearly equal amount of pie.

What we will instead do is explain how this models *Contestant's Row* from The Price Is Right (which we introduced earlier):

`https://www.youtube.com/watch?v=TmKP1a03E2g`

How does the pie game resemble The Price Is Right? And how is it different?

Consider the following game, which is sort of in between:

> **Example 7 (A Number Guessing Game):** Two or more players try to guess a randomly chosen real number between 0 and 100. Each, in turn, offers a guess. If a player repeats a guess made by an earlier player, then they 'steal' the guess and the earlier player is knocked out of the game.
>
> The winning player is whoever guessed closest, without going over. (If all players guess higher than the random number, then everyone loses.)

Note that the numbers 0 and 100 are a little bit arbitrary here – if the random number was chosen, say, between 500 and 1500, then the game would have the same structure.

> **Remark:** Remember that the *real numbers* include more than just the integers: they include fractions, irrational numbers like e and π, arbitrary decimal expansions.... there is a whole zoo of them.
>
> Since there are infinitely many real numbers between 0 and 100, our earlier notions of probability don't quite make sense: the probability that any particular guess is correct is actually *zero*. Instead, we have to talk about the probability that the number is in a fixed *interval*. For example, the probability that a random number in this range is between 23 and 37.4 is
>
> $$\frac{37.4 - 23}{100} = .144 = 14.4\%.$$
>
> To compute the probability as a percent, just subtract the low end of the range from the high end.

The number guessing game models Contestant's Row by assuming that the price of the scuba gear is a random real number between 0 and 100. It is structured similarly to Contestant's Row, but with some important differences:

- We impose a random model on the range of possible prices. As we mentioned earlier, the actual range doesn't really matter – we could pick a random number between 500 and 1200, and the structure of our number guessing game would be the same.

  But not all prices are equally likely. Prices in the middle are somewhat more likely, and prices also tend to end in 00 or at least 0. 99 is also reasonably common.

Moreover, contestants may have different guesses about how much they think scuba gear costs. If a contestant has actually bought scuba gear recently, then they would have a big advantage.

- On the Price Is Right, the prices are always *integers*. The closest you can come to stealing someone else's guess is *to bid one dollar higher*. Savvy contestants do this all the time. But, sometimes, the earlier contestant will have guessed the price *exactly*. (Indeed, they get a $500 bonus for this, paid in cash on the spot.)

- On the Price Is Right, if everyone guesses too high, then they don't all lose. Instead, the guesses are cleared and everyone gets to make another guess.

  (We could have modeled this in our game, but this would make it much more difficult to analyze!)

So how does our number guessing game model our pie game? Suppose for example, that four players made the following guesses in turn: 40, 65, 65, 0. Write x for the random number. Then the outcome of the game is as follows:

- The first player wins if $40 \leq x < 65$. She has a 25% chance of winning.

- The second player had her guess stolen, so she loses no matter what.

- The third player wins if $65 \leq x \leq 100$. She has a 35% chance of winning.

- The third player wins if $0 \leq x < 40$. She has a 40% chance of winning.

Think of 'chance of winning' as being the same as 'proportion of the pie'. Translating the above guesses into the pie game, we get the following sequence of moves:

- The first player takes 60% of the pie (corresponding to $40 \leq x \leq 100$), leaving 40%.

- The second player takes 35% of the pie (corresponding to $65 \leq x \leq 100$), and takes it from the first player. This leaves the first player with 25% of the pie.

- The third player takes all of the second player's pie.

- The fourth player takes the remaining 40% of the pie, corresponding to $0 \leq x < 40$.

8.2 A Complicated Example

Part of *The Price Is Right* consists of spinning the famous **Big Wheel**. It is played twice each show.

If you search Youtube for clips of the big wheel alone, you are likely to find clips where something unusual happened. From our standpoint, typical clips are more interesting. So we refer to 13:00 or 30:00 of the following clip.

Link: The Price Is Right – Typical Episode

Game Description (The Big Wheel – Price Is Right): The **Big Wheel** consists of twenty numbers – 5 through 100 (i.e. five cents through a dollar), in increments of five. Three players compete, and the player who spins the closest to a dollar without going over advances to the Showcase Showdown.

The players spin in order. Each player spins once, and then either keeps the result or elects to spin a second time and add the two results. If the result is higher than $1.00, the player is eliminated immediately. The winner is the player who spins the highest without going over. (If two or more players tie, they advance to a tiebreaker.)

In addition, players earn a bonus if they spin exactly a dollar – but we will ignore this.

Suppose, for example, that the first player spins sixty cents. If she spins again, she is likely to go over. But if she doesn't spin again, she is likely to be beaten by one of the following two players. What should she do?

We will *start* to analyze the question of understanding what each contestant should do in any situation. A complete analysis is rather complicated, and here we will tackle enough of the problem to give a sketch of a complete solution. This will be by backwards induction, and so our first priority is to understand Player 3's strategy.

We first introduce a couple of assumptions:

- The tiebreaker doesn't involve any decision making, and no player has any advantage over any other. So, we may assume that if the game advances to a tiebreaker, then each tied player has equal chances of winning it.
- The winning player advances to the Showcase Showdown, and we assume that contestants mostly care about being this player.

  There is also bonus money available when spinning the Big Wheel. If you get $1.00 exactly (in either one or two spins, or during the tiebreaker), then you get a bonus prize (as well as a 'bonus spin', which can lead to a second, larger, bonus prize). Because these bonus prizes are available, contestants may be willing to assume a little more risk.

  But we will ignore this factor in our analysis.

Player 3's Strategy: This is very easy, and on the show you will even observe that Barker and Carey don't ask the contestants what they want to do.

If you spin more than the two previous contestants, then you win and obviously you should not spin again. This includes the case where both contestants *busted* by spinning more than $1.00 total: the third player then wins by default, and is only spinning to take a crack at the bonus money.

Conversely, if you spin less than one of the two previous contestants, then you should clearly spin again: otherwise you lose.

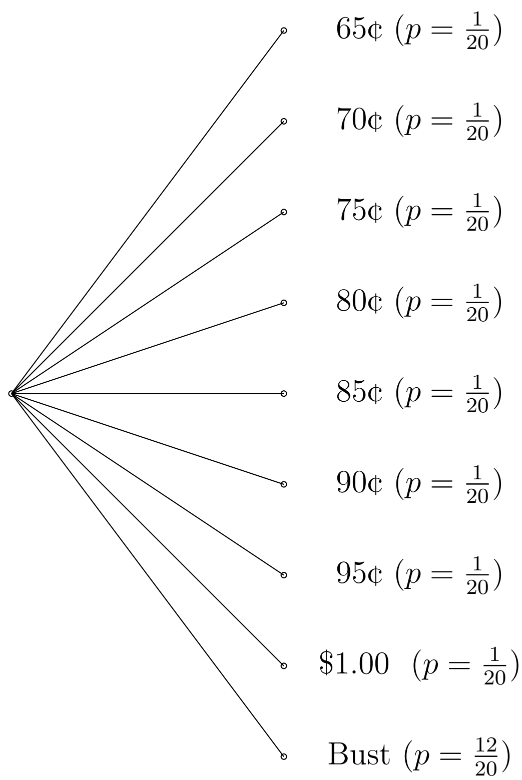
What if you tie? If you and one other player are tied with more than 50 cents, then you have 50-50 odds of winning a tie breaker and less than that of not busting, so you should accept the tie and proceed to the tie-breaker. Conversely, if you and one other player are tied with less than 50 cents, you should spin again. (If you are at exactly 50 cents, then it is a tossup.)

If you are tied with *both* other players, then your odds of winning the tiebreaker are 1 in 3, so you should spin again if the tie is at 65 cents or less, and accept the tie at 70 cents or greater.

Now we consider the following harder question:

Example 8: You are Player 2. The first player spun a total of 55¢, and you spin 60¢ on your second spin. Player 3 is waiting to go after you. Should you keep your 60¢, or spin again?

The following diagram illustrates the possible outcomes, with their probabilities:



With probability $\frac{8}{20}$, your odds of winning will go up, and with probability $\frac{12}{20}$, you will definitely lose. What we have to do now is, for each value between 65¢and $1.00, figure out Player 3's chances of winning – and therefore your chances of winning.

We also have to do this for 60¢, to compute your chances of winning if you *don't spin again*.

We therefore have nine probabilities to compute! We will do them all at once, by solving the following problem:

Example 9: You are Player 3. Suppose that Player 1 is out of the game, and Player 2 has finished with a value of $60 + 5n$, for some integer n with $0 \leq n \leq 8$. What are your odds, and Player 2's odds, of winning?

There are two steps. Firstly, your odds of winning outright, without a tiebreaker, are

$$\frac{8-n}{20} + \frac{11+n}{20} \times \frac{8-n}{20}.$$

Why is this? There are $8 - n$ scores higher than Player 2's total of $60 + 5n$. Your odds of scoring higher on the first spin are therefore $\frac{8-n}{20}$. Your odds of scoring less than Player 2 on the first spin are $\frac{11+n}{20}$, if you do, the odds of overtaking her on the second spin are once again $\frac{8-n}{20}$.

Your odds of forcing a tie are

$$\frac{1}{20} + \frac{11+n}{20} \times \frac{1}{20}.$$

The computation is similar; here $\frac{1}{20}$ is the chance of tying Player 2 on the first spin, and also (if you scored less than her on the first spin) of tying her on the second spin.

Since you have a 50-50 chance of winning a tiebreaker, your total odds of winning are

$$\begin{aligned} &\frac{8-n}{20} + \frac{11+n}{20} \times \frac{8-n}{20} + \frac{1}{2}\left(\frac{1}{20} + \frac{11+n}{20} \times \frac{1}{20}\right) \\ &= \frac{320-40n}{800} + \frac{176-6n-2n^2}{800} + \frac{20}{800} + \frac{11+n}{800} \\ &= \frac{527-45n-2n^2}{800}. \end{aligned}$$

The odds of Player 2 winning are 1 minus this.

The results of the previous computation can be plugged into a calculator or a computer. We make a table, which will illustrate the results and allow us to gut check them. (It is easy to make algebra mistakes when doing computations like this!)

Player 2 Result	n	Player 3 Odds	Player 2 Odds
60¢	0	.659	.341
65¢	1	.600	.400
70¢	2	.536	.464
75¢	3	.468	.532
80¢	4	.394	.606
85¢	5	.315	.685
90¢	6	.231	.769
95¢	7	.142	.858
$1.00	8	.049	.951

Remark: How can we usefully double check the previous table? As Player 2's spin goes higher and higher, her odds of winning improve also. That is a good start!

It might also be helpful to check just a single one of the entries. For example, if Player 2 spins $1.00, what are Player 3's odds of winning? Her odds of forcing a tiebreaker are

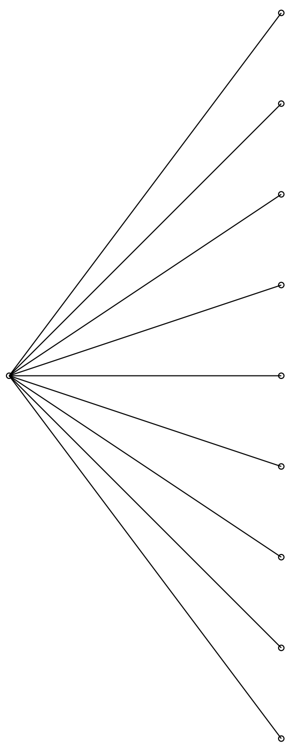
$$\frac{1}{20} + \frac{19}{20} \times \frac{1}{20} = .098,$$

or approximately 10%: she has a 5% chance on the first spin, and another 5% chance on the second spin (if she didn't tie on the first spin).

So her odds of winning are half that, or .049, matching the table.

Making mistakes is human, and it happens to all of us. It is therefore a great idea to double-check your work whenever you get a chance.

We now update our previous probability tree to include the winning probability for each possible outcome:



We're almost done! The probability of winning if you spin again is

$$\frac{1}{20}\times.400+\frac{1}{20}\times.464+\frac{1}{20}\times.532+\frac{1}{20}\times.606+\frac{1}{20}\times.685+\frac{1}{20}\times.769+\frac{1}{20}\times.858+\frac{1}{20}\times.951+\frac{12}{20}\times 0 = .263.$$

26.3%, which is not very good. Alternatively, the probability of winning if you don't spin again is the $n = 0$ entry from the previous table, which is 34.1%. Not great, but better. **So we conclude that you shouldn't spin again.**

Remark: A similar analysis will tell us what Player 2 should do in any situation, where she beats Player 1 on her first spin, and is trying to decide whether to go again.

We can also figure out what Player 2 should do if she *ties* Player 1 on her first spin. This situation is much less good for Player 2: if she stays, then *in the best* case she advances to a tiebreaker (and of course Player 3 might beat her). So, for example, if she ties at 55¢or 60¢, she should spin again. We leave it as an exercise to work out the details!

We figured out Player 2's odds of winning. One can also figure out Player 1 and Player 3's odds of winning: if Player 2 busts, then Player 1 is still in the game and has some chances to win.

And finally, we *outline* how to answer the following question:

Example 10: Figure out the best action for all players, in all situations.

We previously saw how to do this for Player 2 and Player 3. So we now run backwards induction again:

- Suppose, for example, that Player 1 spins 60¢and is trying to decide whether to spin again. We know that Player 2 will hold on any value of 65¢or higher. (We computed that she should hold if she spins 60¢and knocks out Player 1. So if she has a higher spin, then she should certainly also hold then too.) Player 2 will spin again on any value of 55¢or less. We didn't compute what Player 2 would do in the event of a tie, but we could compute that too.

- Since we know how Player 2 will play, we can compute the winning probabilities for all three players if Player 1 keeps her spin of 60¢.

- If Player 1 decides to go again, then she will bust with probability $\frac{12}{20}$, and with probability $\frac{1}{20}$ each she will improve her score to 65¢, 70¢, 75¢, 80¢, 85¢, 90¢, 95¢, or $1.00. For each of those final outcomes, one also computes the winning probabilities for all three players, and for Player 1 in particular.

- Finally, one makes a probability tree for Player 1, just like we did for Player 2, and compares these winning probabilities like we did before. We can then figure out whether it's to Player 1's advantage, on average, to spin again.

The details are complicated, but this is a problem that *we know how to solve*. All we need for a complete solution is the time and willpower to finish.

Or..... a computer. What we described is an *algorithm* for completely solving the question: we break it up into a lot of small steps, and we are guaranteed that following them will eventually yield a complete solution. This is what computers excel at!

This turned out to be worth a research paper. The authors determined that in the first spot, one should spin again with 65 cents or less:

`http://fac.comtech.depaul.edu/rtenorio/Wheel.pdf`

9 Special Topics

In this section we treat some unusual mathematical topics which come up in game shows.

9.1 Divisibility Tests

The following clip illustrates the Price Is Right game of **Hit Me**.

Link: Hit Me – The Price Is Right

Game Description (Hit Me (The Price Is Right)): The contestant plays a game of blackjack against a dealer. As in ordinary blackjack, the objective is to get as many points as possible without going over 21.

The dealer's first card is shown, and then the contestant goes. She is shown six prizes along with six prices, and each price is $n\times$ the actual price of the corresponding item. Behind each price is a card worth n. (If $n = 1$, the contestant gets an *ace* which is worth either 1 or 11, depending on the contestant's preference.) Her aim is to get as close to 21 without going over. She chooses cards one at a time, until she chooses to *stand* and not draw any more cards.

If she gets 21 with exactly two cards, she has a *blackjack* and wins immediately. If she gets over 21, she *busts* and loses immediately. Otherwise, the dealer then goes.

As in ordinary blackjack, the dealer takes two cards at random from a 52-card deck, and then keeps drawing cards until the total is 17 or higher. If the dealer busts, then the contestant wins. Otherwise, the contestant wins if her total beats or (unlike in ordinary blackjack) ties the dealer.

The six cards available to the contestant always include an ace and a ten, and so the contestant always has the opportunity to win the game immediately.

In the clip, the contestant is shown the following prizes and prices: some kind of joint cream for \$5.58; toothpaste for \$14.37; some fragrance for \$64.90; a six-pack of juice for (???? – poor camera work); some calcium supplements for \$76.79; and some denture adhesive for \$27.12. Can you figure out what cards are likely to be behind which prices?

We have to remember the *divisibility tests* from number theory.

Theorem 1 (Divisibility Tests):

- A number is divisible by 2 iff its last digit is.
- A number is divisible by 3 iff the sum of its digits is.
- A number is divisible by 4 iff its last two digits are.

- A number is divisible by 5 iff its last digit is.
- A number is divisible by 6 iff it is divisible by both 2 and 3.
- There are divisibility tests for 7, but it is probably easier to just try dividing by 7 in your head.
- A number is divisible by 8 iff its last three digits are.
- A number is divisible by 9 iff the sum of its digits is.
- A number is divisible by 10 iff its last digit is 0.

Here the word 'iff' is not a misspelling of 'if', but mathematical shorthand for **if and only if**, describing a **necessary and sufficient condition**. For example, if the sum of a number's digits is divisible by 3, then the number is divisible by 3. Since the divisibility tests are 'if and only if', you also know that if the sum of a number's digits is *not* divisible by 3, the number is *not* divisible by 3.

We can use these to figure out what prices are divisible by what.

- 558 is divisible by 2, 3, 6, and 9. Pretty obviously the joint cream is not 62 cents, but it could well be $2.79 and so this one is a little bit tricky to guess.

  So, the card could be any of the ace, two, three, six, or nine, with the ace or the two more likely.

- 1437 is divisible by 3 (only). It looks like the toothpaste is $4.79.
- 6490 is divisible by 2, 5, and 10. If the price of the juice does not end in a ten, and we know that one of the cards is a ten (which it always is), then we know this has to be the ten.
- We have no idea what card the juice hides, because the camera operator bungled their job.
- 7679 is divisible by 7 (only). The cheap way to see this is to eliminate 3 and 9 immediately; it's not even, it's not divisible by 5, so 7 is the only thing that's left unless we believe that the supplements cost this much money.
- 2712 is divisible by 2, 3, 4, 6, and 8. It's difficult to guess the actual price.

We can see if that if you are willing to do some arithmetic in your head, you can do quite well in this game!

9.2 Recursion, Induction, and Gray Codes

Here is a clip of the Price Is Right game of **Bonkers**:

Link: The Price Is Right – Bonkers

> **Game Description (Bonkers (The Price Is Right)):** The contestant is shown a prize whose price is four digits. She is then shown a board with a four digit price for the item. Each digit is wrong, and there are spaces to put paddles above and below each digit.
>
> She must guess whether each digit is too high or too low, by placing paddles in the appropriate location and hitting a button (after which she gets the prize if her guess is right, and buzzed if it is wrong). She has thirty seconds and may guess as many times as she is physically able to.

In the clip, the contestant wins the prize, but she only manages four guesses and wins at the last second. Her strategy leaves much to be improved upon. Here is a contestant who puts on a much better show:

Link: The Price Is Right – Bonkers (Played Well)

We ask: *what's the optimal strategy?*

To answer this question we have to decide what we *mean* by an optimal strategy. There are $2^4 = 16$ possible locations for the paddles. In the absence of any assumptions (can the contestant guess the first digit?), the contestant should simply try as many of them as possible. Since she is racing against the clock, the following guidelines will be helpful:

- She should move as few paddles as possible between guesses. Ideally, she would be able to iterate among all possible guesses moving *only a single paddle at a time.*
- She should move the paddles on the left more frequently than those on the right. They are closer to the buzzer, which means she can move them more quickly.
- The strategy should not be too complicated, since the contestant doesn't have time to pause and think.

You have to start somewhere, and indeed you can start anywhere:

- Step 0. Place all of the paddles somehow. For example, you can place them all on top, although it doesn't really matter.

The real question is, how to cycle through all the possibilities once some initial placement has been made.

It turns out the second bullet point in our strategy consideration is an excellent starting point. Our strategy to go through all possibilities for the four paddles is the following:

- Step 1. Go through all possibilities for the first 3 paddles. Hit the buzzer after each.
- Step 2. Move the fourth paddle.
- Step 3. Go through all possibilities for the first 3 paddles (again). Hit the buzzer after each.

So you only ever have to move the last paddle *once!* This is an excellent strategy – *if* we can figure out how to go through all possibilities for the first 3 paddles. How do we do that?

Let's try this:

- Step 1. Go through all possibilities for the first 2 paddles. Hit the buzzer after each.
- Step 2. Move the third paddle.
- Step 3. Go through all possibilities for the first 2 paddles (again). Hit the buzzer after each.

Ah, but how do we go through all possibilities for the first 2 paddles? You guessed it –

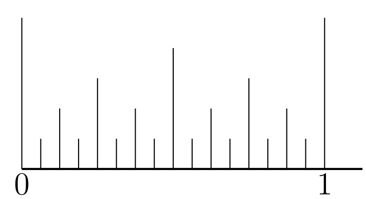
- Step 1. Go through all possibilities for the first paddle. Hit the buzzer after each.
- Step 2. Move the second paddle.
- Step 3. Go through all possibilities for the first paddle. Hit the buzzer after each.

And, at this point, to 'go through all possibilities for the first paddle', you simply do the following: hit the buzzer, move the first paddle, hit the first buzzer again.

This solution produces the following sequence of paddle moves:

$$1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1.$$

This may be a familiar sequence. For example, if you look at an ordinary ruler, it probably has a pattern of tick marks which look something like this:



There are fifteen tick marks between 0 and 1, of four different lengths. If you write down the sequence of their lengths, with 1 being the shortest and 4 the longest, then you will have written down the sequence above!

We have described our strategy in terms of *recursion*. If we know a strategy for 3-paddle Bonkers, then we can deduce a strategy for 4-paddle Bonkers. If we know a strategy for 100-paddle Bonkers, we can deduce a strategy for 101-paddle Bonkers. And so on.

Indeed, the principle of *mathematical induction* tells us that there is no limit to how many paddles we can use:

Theorem 2 (The Principle of Mathematical Induction – for Bonkers):
Suppose that:

- You describe a strategy for playing Bonkers with 1 paddle.
- For each n, you describe a strategy for playing Bonkers with $n+1$ paddles, in terms of n-paddle Bonkers.

Then, this gives you a strategy for any number of paddles.

Mathematical induction is a common technique used in constructing mathematical proofs. Moreover, the sequence of paddle positions produced by our strategy are called **binary Gray codes** and have applications in electrical engineering.

9.3 Inclusion-Exclusion

The *addition rule for counting* stated that if A and B are disjoint sets, then

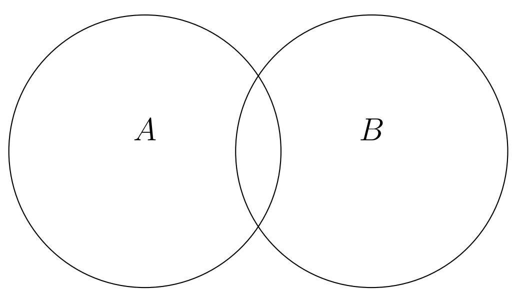
$$N(A \cup B) = N(A) + N(B).$$

The principle of **inclusion and exclusion** is a version of the addition rule that works when the sets are not disjoint. The version for two sets is fairly easy, but the generalization to more sets is complicated.

Inclusion-exclusion for two sets. If A and B are any two sets, then we have

$$N(A \cup B) = N(A) + N(B) - N(A \cap B).$$

This is illustrated by the following Venn diagram. Everything in $A \cap B$ was counted twice – once for A, once for B, so we need to subtract it once to make sure it wasn't double counted.



Note that $N(A \cap B) = 0$ if and only if A and B are disjoint. So this is a generalization of the old addition rule.

Example 3: How many integers between 1 and 100 are divisible by either 2 or 3?

Solution. There are 50 integers in the set divisible by 2, and 33 divisible by 3. (3×1 through 3×33.) An integer is divisible by both 2 and 3 if and only if it is divisible by 6, and there are 16 of these.

So the count is

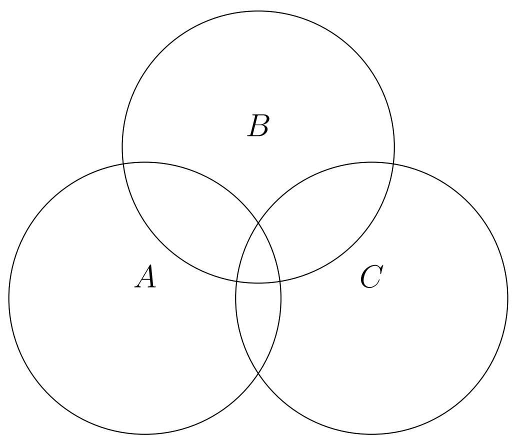
$$50 + 33 - 16 = 67.$$

You can check it! Another way to count the same: an integer n is divisible by 2 or 3 if its remainder after division by 6 is 2, 3, 4, or 0. So, four out of every six. In the first 96 integers, there are 16 groups of six and exactly 64 integers in this range that we want to count. Finally, out of the last four integers (97, 98, 99, and 100) there are three we want to count. So 67 total.

Inclusion-exclusion for three sets. If we have *three* sets A, B, and C, then the rule is

$$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C).$$

Here is a Venn diagram.



The formula is probably not obvious, but you can check it from the diagram. For example, if an element is in A, not B, and not C, then the number of times it is counted is

$$1 + 0 + 0 - 0 - 0 - 0 + 0 = 1.$$

The terms are in the same order as in the formula above. It appears in only the $N(A)$ term on the right!

If an element is in A and B, but not C, then the number of times it is counted is

$$1 + 1 + 0 - 1 - 0 - 0 + 0 = 1,$$

and if an element is in all three then the number of times it is counted is

$$1 + 1 + 1 - 1 - 1 - 1 + 1 = 1.$$

The other cases are exactly analogous. So, each element is counted exactly once, unless it is in none of the sets.

The general **principle of inclusion-exclusion** says that this process works with any finite number of sets. Given sets A_1, A_2, through A_k,

- Consider each of the sets, and sum all of their sizes. Then,
- Consider the double overlaps, and sum the sizes of all the double overlaps. Subtract these. Then,
- Consider the triple overlaps, and sum all their sizes. Add these. Then,
- Consider the quadruple overlaps, and sum all their sizes. Subtract these. Then, ...
- And so on. Keep going, alternating addition and subtraction until you've counted the overlap of all of the sets. (You will count it negative if there are an even number of sets, and positive if there are an odd number of sets.)

9.3.1 The umbrella problem

We consider the following problem:

The Umbrella Problem. One hundred guests attend a party. It is raining, and they all bring umbrellas to the party. All of their umbrellas are different from each other.

At the end of the party, the host hands umbrellas back to the guests at random. What is the probability that *nobody* gets their own umbrella back?

This was asked as a probability question, but we will reframe it as a counting question. We consider the number of ways to give umbrellas out to the party guests. This is essentially the same as a *permutation* of the umbrellas, so there are 100! possibilities. This number is equal to

93326215443944152681699238856266700490715968264381621468592963895217599993229915608941463976156518286253697920827223758251185210916864000000000000000000000000.

We just have to count how many of them don't involve giving anyone their own umbrella! No problem, right?

We'll do the opposite count, and count how many involve giving *at least* one person their own umbrella. We do this using inclusion-exclusion. A_1 is the set of ways to distribute the umbrellas, with person #1 getting their own umbrella. A_2 is the set of ways to distribute the umbrellas, with person #2 getting their own umbrella. And so on, there are a hundred sets.

Note that, when counting A_1, we don't have to worry about whether other people get their umbrella or not! This is much easier. If we needed to count the ways to distribute the umbrellas, with person #1 and *only person #1* getting their umbrella, this would be harder. (It would be like the mani problem we're solving here.) The nice thing is that we have formulated the problem so we never have to worry about who *doesn't* get their own umbrella, only who *does*.

We count $N(A_1 \cup A_2 \cup \cdots \cup A_{100})$ and then subtract it from that big number. We do this using inclusion-exclusion.

- $N(A_1)$ is just 99!. We give person #1 their own umbrella, and distribute the other umbrellas any which way.

  $N(A_2)$ through $N(A_{100})$ are also each 99!, for the same reason.

  So, the total number added in this step is

  $$99! \times 100 = 100!.$$

- $N(A_1 \cup A_2)$ is 98!. We give the first two people their own umbrella, and distribute the others however.

  How many sets are there like this? Exactly the number of ways to choose two people out of 100, which is $C(100, 2) = \frac{100!}{98!2!}$. So the total number subtracted in this step is

  $$98! \times \frac{100!}{98!2!} = \frac{100!}{2!}.$$

- $N(A_1 \cup A_2 \cup A_3)$ is 97!, as before. The number of sets like this is $C(100, 3) = \frac{100!}{97!3!}$, and the number *added* in this step is

  $$97! \times \frac{100!}{97!3!} = \frac{100!}{3!}.$$

- The pattern continues. For the four-fold intersections we subtract

  $$96! \times \frac{100!}{96!4!} = \frac{100!}{4!},$$

  and then we add $\frac{100!}{5!}$, subtract $\frac{100!}{6!}$, and so on. The very last step is subtracting $\frac{100!}{100!}$ – the one way in which we can give everyone their correct umbrella!

So the total number of ways to distribute the umbrellas with *at least one person* getting their umbrella is

$$100! - \frac{100!}{2!} + \frac{100!}{3!} - \frac{100!}{4!} + \cdots - \frac{100!}{100!}.$$

The number of ways to distribute the umbrellas with *nobody* getting their umbrella is 100! minus this, or

$$100! - 100! + \frac{100!}{2!} - \frac{100!}{3!} + \frac{100!}{4!} - \cdots + \frac{100!}{100!},$$

which we can rewrite as

$$100!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots + \frac{1}{100!}\right),$$

and so upon dividing by 100! (which was the *total* number of ways to distribute the umbrellas we see that the probability that no one gets their umbrella is

$$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots + \frac{1}{100!}.$$

This is a pretty good answer! But we can do better if we know some calculus. Calculus tells us that the **Taylor series expansion** for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots ,$$

and so plugging in e^{-1} we get

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots + \frac{1}{100!} - \frac{1}{101!} + \cdots$$

If we truncate after $\frac{1}{100!}$ then we get **exactly our umbrella probability**, and also by the **alternating series test** we make an error less than $\frac{1}{101!}$, which is very **VERY** small – less than one over the **GIANT** number above, and so the probability that no one gets their umbrella is, within an error bounded by $\frac{1}{101!}$, equal to

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots = 0.3678794411714423215955237701614608674 5\cdots$$

9.3.2 Switcheroo

The game **Switcheroo** is illustrated in the following clip.

`https://www.youtube.com/watch?v=nvSMVAuGpAE`

Game Description (Switcheroo – The Price Is Right): The contestant is shown five prizes – four very small prizes and a car. The contestant is also shown the price of each with the tens digits removed. The tens digits of the prizes are all different from each other, and the contestant is given five blocks with the tens digits written on them.

The contestant must match the blocks to the removed tens digits. She has thirty seconds to do this, after which she is shown how many prizes she has matched correctly (but not *which* prizes she has matched correctly). She may then switch around the blocks if she likes. She wins all prizes which she prices correctly.

We will ask the following question: **A contestant has no idea what any of the prices are, but otherwise plays optimally. With what probability does she win the car?**

Since she has no idea what the prices are, she just places the blocks randomly in the first round (any placement is as good as any other). Her choice of strategy depends on how many she gets right on the first round:

- If she gets 2, 3, or 5 correct then this is better than expected and she should stick with her guess. Her probability of winning the car is $\frac{2}{5}$, $\frac{3}{5}$, or 1 respectively.

  (Note that there is no way to get exactly four correct, or equivalently, exactly one wrong. If one prize's block is in another prize's slot, then her guesses for *both* prizes must be wrong.)

- If she gets 1 correct then she is indifferent to switching or leaving everything in place. There is a $\frac{1}{5}$ probability that her one correct item is the car, and a $\frac{4}{5}$ probability that it is one of the other items – in which case each of the remaining numbers is equally likely to be the correct price for the car.

- If she gets none correct, then she should switch. This is *better* than getting exactly one correct because she got some reliable information: her guess for the car is wrong. So she picks one of the other numbers and wins with probability $\frac{1}{4}$.

So we have to figure out the probabilities of each of these outcomes on the first round! There are $5! = 120$ ways to place the blocks, and so all of these probabilities will be fractions with 120 in the denominator. In our analysis we will label the prizes A, B, C, D, and E.

- *Five right.* There is exactly one way.

- *Three right.* First we ask: in how many ways can the contestant get A, B, and C right and D and E wrong? One: ABCED.

  So the number of ways to get exactly three right is $C(5,3) = 10$ – the number of subsets of three of the five prizes.

- *Two right.* First we ask: in how many ways can the contestant get A and B right and C, D, and E wrong? There are two: ABDEC and ABECD.

  So the number of ways to get exactly three right is $C(5,3) \times 2 = 10 \times 2 = 20$: there are ten ways in which to choose which subset she gets right, and for each, two ways to screw the rest up.

- *None right.* This is the umbrella problem!! The answer is

$$5! - 5! + \frac{5}{2!} - \frac{5}{3!} + \frac{5}{4!} - \frac{5}{5!} = 44.$$

- *One right.* By process of elimination,

$$120 - (1 + 10 + 20 + 44) = 120 - 75 = 45.$$

  Alternatively, there are five ways to choose one prize to get right, and

$$4! - 4! + \frac{4!}{2!} - \frac{4!}{3!} + \frac{4!}{4!} = 9$$

  ways to mix up the rest, and $5 \times 9 = 45$.

So her probability of winning in the end is

$$\frac{1}{120} \times 1 + \frac{10}{120} \times \frac{3}{5} + \frac{20}{120} \times \frac{2}{5} + \frac{45}{120} \times \frac{1}{5} + \frac{44}{120} \times \frac{1}{4} = \frac{7}{24}.$$

10 Review

Here we briefly review the main ideas of the course and propose some sample questions relevant to each. (This is *not* a comprehensive listing of every topic covered.)

Probability and counting We formally defined probability in terms of *sample spaces* and *events*. These were subject to the *addition* and *multiplication* rules. The former said that probabilities for disjoint events add; the latter that probabilities for sequential events multiply.

We also supplemented this material with some material on permuations and combinations – you should remember the formulas for those. These help with counting sizes of events and sample spaces.

Good sample questions involve coins, dice, and cards. You are dealt two cards. What is the probability that (1) they are of the same suit; (2) they are each ten or higher; (3) they could possibly fit into a five-card straight; etc. (Compose your own!) You toss three dice. What is the probability the sum is odd? Even? At least twelve? Bigger than 14 or smaller than 5? You flip ten coins. What is the probability that they all come up heads? Half of them?

Probability questions also came up in various game shows. On The Price Is Right, Rat Race, Let 'Em Roll, Squeeze Play, Switcheroo, 3 Strikes, Spelling Bee, and Plinko (among others) provide lots of probability questions. Watch an episode, start to finish, and see what you can come up with. You can also come up with interesing probability questions watching Deal or No Deal: what is the probability that the contestant will have eliminated the two most valuable briefcase by the time the bank's first offer comes in?

Finally, poker was an excellent source of probability questions.

Expected value. Make sure you understand how expected value works. Expected value comes up in poker, in game shows like Let's Make a Deal and Deal or No Deal, and pretty much every scenario where probability is relevant. Here again you can compose your own questions. You toss three dice, and get a dollar for every six you roll. Alternatively, you get a dollar if at least two dice are the same. What is the value of playing such a game?

Remember the rule of *linearity of expectation*, and review its applications. The idea is that *expected values add*. For example, if you get a dollar for every six you roll in three dice, you do *not* need to compute the probabilities of rolling zero, one, two, or three sixes. Just compute the expected value of one die, and multiply by three.

We also introduced conditional probability. Make sure you understand the definition and why it is true. Review the Monty Hall Problem and its variants (and related games like

Barker's Markers). And be sure you understand how to use Bayes's theorem, either in the form of the formula or in terms of reasoning via probability trees.

Strategic game theory. We covered this *very* lightly (it is easily worth an entire undergraduate course). Understand how these are set up and how a payoff matrix corresponds to a game. You should also be able to find the Nash equlibrium in a game with two choices for each player. This might be a pure or mixed strategy. (Try finding the Nash equilibria of all the games we discussed, and thinking up your own games.)

Backwards induction. There were no exercises on this, and any exam problems on this will be relatively easy. Note also that we did some backwards induction problems before introducing it per se. For example, Punch-a-Bunch is very much backwards induction.

11 Project Ideas

Part of the course requirements is a term project: study a game or game show in depth, write a paper analyzing it, and give a presentation in class.

Here are some ideas. Of course, feel free to come up with your own.

- **Deal or No Deal**: This is an easy to understand game from the contestants' point of view. What about the producers? How does the bank determine its offers?

  Watch a bunch of episodes of the show and write down what happens. Attempt to determine some sort of formula that predicts what the bank will offer.

- **Press Your Luck**: One interesting project would be to investigate the patterns behind the show, just as Michael Larson did. Watch old YouTube videos, and hit freeze frame a lot! Try to describe the patterns, and see if you could win $100,000 too.

- **Switcheroo**: Here is a fascinating, and deep Price Is Right game:

  `https://www.youtube.com/watch?v=nvSMVAuGpAE`

  Try to figure out the optimal strategy. You will have to assume that the contestant has *some* idea how much the small prizes cost, but very imperfect information.

- **Race Game**: A somewhat easier Price is Right game. Here is a clip:

  `https://www.youtube.com/watch?v=CkqZkqeNyKU`

  You might try to figure out the best strategy, assuming the contestant has no idea how much the prizes cost.

  This is somewhat similar to the game *Mastermind* (see the Wikipedia page). But don't neglect the fact that some prizes are closer to the lever than others!

- **Poker**: If you enjoyed the poker discussion, you might want to dig deeper. I recommend reading at least the first of Harrington's books, watching some poker tournaments online, and then trying to analyze what happened.