Algebraic number theory (Spring 2013), Homework 1

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Due Friday, February 1

Please do at least one of the cubic field integral basis calculations, if you do nothing else. Questions 2 and 3 can be skipped without loss of continuity, but are great questions for prospective algebraists or algebraic geometers.

1. (3 points) Is $\frac{3+2\sqrt{6}}{1-\sqrt{6}}$ an algebraic integer?

2. (5 points) If the integral domain A is integrally closed, then so is the polynomial ring $A[t]$.

3. (5 points) In the polynomial ring $A = \mathbb{Q}[X, Y]$, consider the principal ideal $\mathfrak{p} = (X^2 - Y^3)$. Show that $\mathfrak{p}$ is a prime ideal, but $A/\mathfrak{p}$ is not integrally closed.

4. (8 points) Find the discrimimant of, and an integral basis for, $\mathbb{Q}(2^{1/3})$.

5. (10 points) Find the discriminant of, and an integral basis for, $\mathbb{Q}(\theta)$ where $\theta^3 - \theta - 4 = 0$.

6. (10 points) Find the discriminant of, and an integral basis for, $\mathbb{Q}(\theta)$ where $\theta^7 - 3 = 0$.

7. (*5 points) Prove that the discriminant of an algebraic number field is always congruent to 0 or 1 modulo 4.

   Hint: the determinant of an integral basis is a sum of terms, each prefixed by a positive or a negative sign. Writing P and N for the positive and negative terms respectively, the discriminant is $(P - N)^2 = (P + N)^2 - 4PN$. The solution is most naturally explained using a little Galois theory.

8. (5 points) In the ring $\mathbb{Z}[\sqrt{-5}]$, verify that

$$(2) = (2, 1 + \sqrt{-5})(2, 1 - \sqrt{-5})$$

$$(3) = (3, 1 + \sqrt{-5})(3, 1 - \sqrt{-5})$$

$$(1 + \sqrt{-5}) = (2, 1 + \sqrt{-5})(3, 1 + \sqrt{-5})$$

$$(1 - \sqrt{-5}) = (2, 1 - \sqrt{-5})(3, 1 - \sqrt{-5}).$$

   In each of the four factorizations above, are the two ideals on the right different from each other?

9. (5 points) Verify that the equation $a^2 + 5b^2 = 2$ does not have any solutions over $\mathbb{Q}$.