**Comprehensive exam syllabus for Math 788p, Algebraic Number Theory (Spring 2013)**

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*Note: Probably this will be combined with Michael Filaseta's comp for Fall 2012. He covered the basics, which you should be sure to learn very well. The list of topics here is mostly new, but there might be some overlaps: that means we both agree it's really important.*

1. Know the definitions of a *ring of integers* and a *discriminant*, and be able to compute them. Be able to compute rings of integers and discriminants for arbitrary number fields. (This can get *very* messy, and you will not be asked extremely messy problems on the exam.)

2. Understand that unique factorization may or may not hold in rings of integers. Be able to prove it in typical examples, such as $\mathbb{Z}[i]$, and be able to disprove it in typical examples, such as $\mathbb{Z}[\sqrt{-5}]$.

3. Be familiar with the $p$-adic numbers $\mathbb{Z}_p$ and $\mathbb{Q}_p$. Understand and work with definitions in terms of an inverse limit, or in terms of a completion, and prove that they are the same. Understand and be able to prove the ideal structure of $\mathbb{Z}_p$. Prove that $\mathbb{Z}$ and $\mathbb{Z}_{(p)}$ embed into $Z_p$, and that $\mathbb{Q}$ embeds into $\mathbb{Q}_{(p)}$.

4. Solve equations in $\mathbb{Z}_p$ and $\mathbb{Q}_p$. Be able to represent fractions, square roots, etc. as $p$-adic numbers where appropriate. When no such representations exist, be able to prove it. Know the statement of Hensel's lemma.

5. Be able to describe the deomposition of prime ideals in number fields. Know the Chinese remainder theorem; be able to compute norms of ideals and elements. Know the $efg$ theorem, and be able to enumerate the possible splittings of primes in extensions (including the Galois case, covered later in the semester).

6. Know the theorem relating factorization of polynomials mod $p$ to factorization of $p$ in rings of integers of number fields. Use it to compute splitting of prime ideals in extensions.

7. Be able to define fractional ideals and the class group. Compute the class group in appropriate special cases. (You will not be asked to memorize the Minkowski bound.)

8. Know the statement of Dirichlet's unit theorem. Be able to find the torsion part of the unit group, and be able to use the theorem to compute the rank.

9. Know the definitions of the decomposition and inertia groups. Know what their sizes are, and understand why. Be able to prove elementary consequences, for example as seen in the homework.

10. Know the definition of the Artin symbol, be able to prove its basic properties, and know how to compute it in quadratic fields, and in cyclotomic fields and their subfields. Understand the relation between the Artin symbol and Galois groups of polynomials; be able to verify (for example) that individual number fields have Galois group $S_n$ based on the Artin symbol.