The Geometry of Numbers (Spring 2014): Homework 4

Frank Thorne, thorne@math.sc.edu

Asterisks indicate problems representative of what might appear on the comprehensive exam. Plusses indicate problems whose solutions will likely involve background beyond what has been taught here and in 701/702.

1. (* 5 points) Verify directly, via brute force, that the discriminant of a binary cubic form is $SL_2(\mathbb{Z})$-invariant.

   (You are welcome to outsource arithmetic, etc. to Sage or other software, but please use it only for basic algebra and do not call any highbrow routines.)

2. (* 3 points) Let $u^3 + a_2u^2v + a_3uv^2 + v^3$ be a binary cubic form with first and last coefficients 1. Prove that its discriminant is equal to the polynomial discriminant obtained by setting either u or v equal to 1.

3. (* 10 points) Do the exercise spelled out on p. 23.4 of the lecture notes, relating discriminants of forms to discriminants of polynomials.

4. (* 5 points) Carry out the details of the computation given on p. 23.5 of the lecture notes.

5. (* 10 points) Describe what the Delone-Faddeev correspondence says over (some or all of) the following fields: $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{F}_p$, $\mathbb{Q}$, $\mathbb{C}(t)$, $\mathbb{Q}_p(t)$. Describe both sides of the correspondence, and explain what conclusions Delone-Faddeev allows you to draw, in case any of them are nontrivial.

6. (5 points) Prove that there are $\frac{1}{3}(p^2 - 1)(p^2 - p)$ irreducible binary cubic forms over $\mathbb{F}_p$. (Hint: use Delone-Faddeev.)

7. (12 points) Formulate the natural generalization of Delone-Faddeev to quartic forms and fields, and illustrate by counterexample that it does not hold.

8. (* 5 points) Write down some cubic rings (including some of the weird ones) and compute their discriminants.

9. (* 15 points) Work out several explicit examples of the Delone-Faddeev correspondence over $\mathbb{Z}$. Your examples should include reducible and irreducible binary cubic forms, including a binary cubic form which factors as the product of a linear times a quadratic; integral domains, rings with zero divisors but no nilpotents, and rings with nilpotents. Compute the relevant discriminants, and summarize your conclusionsl.

10. (* 10 points) Is the following true or false?

    Consider the cubic ring $\mathbb{Z}[\alpha]$, where $\alpha^3 + b\alpha^2 + c\alpha + d = 0$. Then, the corresponding cubic form is $u^3 + bu^2v + cuv^2 + dv^3$.

    If this is false (or imprecisely stated), find a better version of this statement if possible.