**The Geometry of Numbers (Spring 2014): Homework 2**

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Asterisks indicate problems representative of what might appear on the comprehensive exam. Plusses indicate problems whose solutions will likely involve background beyond what has been taught here and in 701/702.

1. (* 5 points) Let $ax^2 + bxy + cy^2$ be a quadratic form of discriminant $D \neq 0$. Prove that it is positive definite if and only if its discriminant $D > 0$ and $a > 0$. In addition, describe what happens if $D = 0$.

2. (* 5 points) Can a quadratic form be indefinite over $\mathbb{R}$, but only represent positive integers when $x, y \in \mathbb{Z}$?

3. (* 5 points) Prove that the action of $GL_2(\mathbb{Z})$ defined in lecture does *not* define a *left* action on binary quadratic forms.

   In other words, find $g, g'$ and $f$ for which (if a left action was defined) we would have $g(g'(f)) \neq (gg')(f)$.

4. (* 5 points) Prove *directly* (i.e. do not quote the reduction theorem) that the quadratic forms $x^2 + 5y^2$ and $2x^2 + 2xy + 3y^2$ are not equivalent.

5. (10 points) Filling in all of the remaining details only sketched in class, present a complete proof that every primitive positive definite quadratic form is properly equivalent to a unique reduced form.

6. (2 points each, up to 10) Compute $h(D)$ for $D = -7. - 8, -163, -67, \cdots$.

7. (* 5 points) Find some $D$ for which $h(D) > 5$.

   (Hint: Consider using Dirichlet's class number formula to guess how big $h(D)$ will be, then use the reduction theory.)

8. (* 10 points total, includes partial credit) Compute the automorphism group of an arbitrary positive definite quadratic form.

   (At least do $x^2 + xy + y^2$, that is the interesting one. Also do $x^2 + ny^2$ for $n > 1$.)

9. (15 points) Give a complete proof of the formula for $r_D(n)$, possibly with conditions like $D$ is odd, coprime to $n$, etc. The Cox exercise passed out in class gives an excellent blueprint, or roll your own.

10. (* 10 points) Find the fundamental units for all positive discriminants $D < 14$, and write down the corresponding automorphism group.

11. (10 points) In a page or so, describe what you learned from **Nathan Ilten**'s colloquium lecture, along with some additional topics (if any) that his lecture led you to want to learn.

12. You will also get credit for **any and all** related exercises in Granville's notes or Cox's book which you hand in. **All** of them are interesting and highly relevant to this course and to number theory in general.

13. (**Bonus**) Read Granville's notes, and submit a list of typos, corrections, or mistakes (if you find any). I will submit a list to him with names of submitters included.