ON A PRINCIPLE OF LIPSCHITZ

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1. It is often desirable to be able to approximate to the number of points with integral coordinates in a closed bounded region $\mathfrak{R}$ in n dimensional space by means of the volume $V(\mathfrak{R})$ of $\mathfrak{R}$. The position is very simple if $\mathfrak{R}$ depends on one parameter X, and is obtained from a fixed region $\mathfrak{R}_1$ by uniform dilatation about the origin with linear ratio of dilatation X. Then it is immediate that the number of points with integral coordinates in $\mathfrak{R}$ is

$$V_1 X^n + O(X^{n-1})$$

as $X \to \infty$, where V_1 is the volume of $\mathfrak{R}_1$. This, however, is a very special situation. A simple estimate for $N(\mathfrak{R}) - V(\mathfrak{R})$, where $N(\mathfrak{R})$ is the number of points with integral coordinates in $\mathfrak{R}$, is easily given in the two dimensional case. We have†

$$|N(\mathfrak{R}) - V(\mathfrak{R})| < 4(L+1),$$

where L is the length of the boundary of $\mathfrak{R}$, assumed to be a rectifiable curve. In more than two dimensions, I know only of a discussion in Bachmann's *Analytische Zahlentheorie* (436-444), though there may well be other work bearing on the question. Bachmann enunciates a principle which he attributes to Lipschitz‡, though in fact Bachmann's formulation is more explicit than that of Lipschitz. The principle is that, in n dimensional space,

$$|N(\mathfrak{R}) - V(\mathfrak{R})| < \Theta Q,$$

where Q is the greatest $n-1$ dimensional volume of any of the regions into which $\mathfrak{R}$ projects on the n coordinate spaces $x_i = 0$, and Θ is bounded, presumably depending only on n if $\mathfrak{R}$ satisfies suitable conditions. The principle is expressed slightly differently, in the form that the number of points of a cubical lattice of side s in $\mathfrak{R}$ is

$$V(\mathfrak{R})(1/s)^n + \Theta Q (1/s)^{n-1}.$$

If $\mathfrak{R}$ is fixed, and $s \to 0$, this is asymptotically the same as the result first mentioned. But the most interesting application§ is precisely to a case in which $\mathfrak{R}$ is not fixed, but itself varies with s.

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† See, for example, Landau, *Vorlesungen über Zahlentheorie*, 2, 186. Steinhaus, in *Colloq. Math.*, 1 (1947), 1–5, has proved that the result holds with L on the right, provided $L > 1$.

‡ *Monatsber. der Berliner Academie*, 1865, 174 *et seq.*

§ To the average class-number of quadratic forms (Bachmann, *loc. cit.*, 450–459).

The "principle of Lipschitz", as formulated above, is plainly not valid, even for the simplest kind of region. The example of a small sphere round the origin shows that the right-hand side must not be allowed to be less than 1. Even if we amend the statement by reading $Q+1$ in place of Q, it is still false; for in the case of a long thin cylinder round one of the coordinate axes, $N(\mathfrak{R})$ may be arbitrarily large, and $V(\mathfrak{R})$ and Q both arbitrarily small. It seems necessary to introduce into the estimate not only the $n-1$ dimensional projections of $\mathfrak{R}$ but also all the m dimensional projections of $\mathfrak{R}$ on the spaces obtained by equating any $n-m$ of the coordinates to zero.

2. In order to obtain a simple result, we impose the following somewhat restrictive conditions on $\mathfrak{R}$, which is supposed throughout to be a closed and bounded set of points.

I. *Any line parallel to one of the n coordinate axes intersects $\mathfrak{R}$ in a set of points which, if not empty, consists of at most h intervals.*

II. *The same is true (with m in place of n) for any of the m dimensional regions obtained by projecting $\mathfrak{R}$ on one of the coordinate spaces defined by equating a selection of $n-m$ of the coordinates to zero; and this condition is satisfied for all m from 1 to $n-1$.*

THEOREM. *If R satisfies the conditions I and II, then*

$$|N(\mathfrak{R})-V(\mathfrak{R})| \leqslant \sum_{m=0}^{n-1} h^{n-m} V_m,$$

where V_m is the sum of the m dimensional volumes of the projections of $\mathfrak{R}$ on the various coordinate spaces obtained by equating any $n-m$ coordinates to zero, and $V_0 = 1$ by convention.

The various volumes mentioned can be understood as Lebesgue measures, which exist because all the sets are closed and bounded.

In applications to number-theory*, we are concerned with a region consisting of all points $(x_1, \ldots, x_n)$ which satisfy various algebraic inequalities

$$F_i(x_1, \ldots, x_n) \geqslant 0 \quad (i = 1, 2, \ldots, k),$$

where F_i is a polynomial with real coefficients, whose degree is bounded, say by l. It is assumed that the inequalities are such as to imply that $\mathfrak{R}$ is bounded. Under these circumstances, every m dimensional projection of $\mathfrak{R}$ on a coordinate space is also defined by a finite number of algebraic

* See the two following papers in this *Journal*, with the title "On the class-number of binary cubic forms".

inequalities. Both the number of these inequalities and their degrees are bounded in terms of n, k, l. The region $\mathcal{R}$ therefore satisfies the conditions I and II, for a value of h depending only on n, k, l. It follows from the theorem that

$$|N(\mathcal{R})-V(\mathcal{R})|<C\max(\bar{V},1),$$

where C depends only on n, k, l, and $\bar{V}$ is the greatest m dimensional volume of any projection of $\mathcal{R}$ on a coordinate space, m taking all values from 1 to $n-1$. The essential point is that C is independent of the coefficients of the polynomials F_i.

3. *Proof of the Theorem**. Let $f(x_1, \ldots, x_n)$ be the characteristic function of the closed and bounded set $\mathcal{R}$; this is a measurable function, since R is closed. All summations and integrations are extended over some large cube containing $\mathcal{R}$, and variables of summation are integers. If $i_1<i_2<\ldots<i_m$ is any selection of numbers from 1 2 $\ldots$, n, we denote by $f(x_{i_1}, \ldots, x_{i_m})$ the function which is 1 or 0 according as there do or do not exist values of the remaining variables which make $(x_1, \ldots, x_n)$ a point of $\mathcal{R}$. Thus $f(x_{i_1}, \ldots, x_{i_m})$ is the characteristic function of the m dimensional set obtained by projecting $\mathcal{R}$ on the coordinate space in which the remaining $n-m$ coordinates are zero. Such a function is also a measurable function of the variables occurring in it.

By a special case of Fubini's theorem, the measure of $\mathcal{R}$ is given by a repeated integral:

$$V(\mathcal{R})=\int dx_1\int dx_2\ldots\int f(x_1, \ldots, x_n)\,dx_n,$$

and similarly for the measures of the various projections of $\mathcal{R}$.

With the notation just introduced, the explicit formulation of the theorem becomes:

$$(1)\quad \begin{cases} \left|\int dx_1\ldots\int f(x_1, \ldots, x_n)\,dx_n-\sum\limits_{x_1}\ldots\sum\limits_{x_n} f(x_1, \ldots\ x_n)\right| \\ \qquad\qquad \leqslant \sum\limits_{m=0}^{n-1} h^{n-m}\sum\limits_{i_1<\ldots<i_m}\int dx_{i_1}\ldots\int f(x_{i_1}, \ldots, x_{i_m})\,dx_{i_m}; \end{cases}$$

where, in the case $m=0$, the value 1 is to be ascribed to the meaningless inner sum on the right.

When $n=1$, the one dimensional set $\mathcal{R}$ consists, by the hypothesis I, of at most h intervals, and the result is then obvious, since it reduces to

$$\left|\int f(x)\,dx-\sum_x f(x)\right|\leqslant h.$$

* I am greatly indebted to Mr. H. Kestelman for helpful suggestions and advice in connection with this proof.

The general result is now proved by induction on n, assuming the truth for $n-1$. The intersection of the set $\mathfrak{R}$ with any space $x_1 = \xi$ satisfies the conditions of the $n-1$ dimensional theorem, and so for any particular value of x_1 the inductive hypothesis gives

$$\left|\int dx_2 \ldots \int f(x_1, \ldots, x_n)\, dx_n - \sum_{x_2} \ldots \sum_{x_n} f(x_1, \ldots, x_n)\right|$$
$$\leqslant \sum_{r=0}^{n-2} h^{n-1-r} \sum_{\substack{i_1 < \ldots < i_r \\ i_1 \geqslant 2}} \int dx_{i_1} \ldots \int f(x_1, x_{i_1}, \ldots, x_{i_r})\, dx_{i_r}.$$

Integrating the expression inside the modulus sign with respect to x_1, we obtain

$$\left|\int dx_1 \ldots \int f(x_1, \ldots, x_n)\, dx_n - \sum_{x_2} \ldots \sum_{x_n} \int f(x_1, \ldots, x_n)\, dx_1\right|$$
$$\leqslant \sum_{r=0}^{n-2} h^{n-1-r} \sum_{\substack{i_1 < \ldots < i_r \\ i_1 \geqslant 2}} \int dx_1 \int dx_{i_1} \ldots \int f(x_1, x_{i_1}, \ldots, x_{i_r})\, dx_{i_r}.$$

Replacing $r+1$ by m and making a slight change of notation, this is

$$\sum_{m=1}^{n-1} h^{n-m} \sum_{\substack{j_1 < \ldots < j_m \\ j_1 = 1}} \int dx_{j_1} \ldots \int f(x_{j_1}, \ldots, x_{j_m})\, dx_{j_m}. \tag{2}$$

Also by the one dimensional case we have, for any particular values of $x_2, \ldots, x_n$,

$$\left|\int f(x_1, \ldots, x_n)\, dx_1 - \sum_{x_1} f(x_1, \ldots, x_n)\right| \leqslant h f(x_2, \ldots, x_n),$$

on recalling the meaning assigned to $f(x_2, \ldots, x_n)$. Summing over $x_2, \ldots, x_n$, and applying the $n-1$ dimensional result, we obtain

$$\left|\sum_{x_2} \ldots \sum_{x_n} \int f(x_1, \ldots, x_n)\, dx_1 - \sum_{x_1} \ldots \sum_{x_n} f(x_1, \ldots, x_n)\right|$$
$$\leqslant h \sum_{x_2} \ldots \sum_{x_n} f(x_2, \ldots, x_n)$$
$$\leqslant h \int dx_2 \ldots \int f(x_2, \ldots, x_n)\, dx_n$$
$$+ h \sum_{m=0}^{n-2} h^{n-1-m} \sum_{\substack{i_1 < \ldots < i_m \\ i_1 \geqslant 2}} \int dx_{i_1} \ldots \int f(x_{i_1}, \ldots, x_{i_m})\, dx_{i_m}$$
$$= \sum_{m=0}^{n-1} h^{n-m} \sum_{\substack{i_1 < \ldots < i_m \\ i_1 \geqslant 2}} \int dx_{i_1} \ldots \int f(x_{i_1}, \ldots, x_{i_m})\, dx_{i_m}. \tag{3}$$

The sum of the expressions (2) and (3) is an upper estimate for the left-hand side of (1). Adding these two expressions together, we obtain from (3) the term h^n when $m = 0$, and for $1 \leqslant m \leqslant n-1$ we obtain

$$\sum_{i_1 < \ldots < i_m} \int dx_{i_1} \ldots \int f(x_{i_1}, \ldots, x_{i_m})\, dx_{i_m}.$$

This gives the right-hand side of (1), and so proves the result.

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ON THE CLASS-NUMBER OF BINARY CUBIC FORMS (I)

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1. The arithmetical theory of binary cubic forms with integral coefficients was founded by Eisenstein, and further contributions were made by Arndt, Hermite and others†. Two such forms are said to be equivalent if one can be transformed into the other by a linear substitution with integral coefficients and determinant ± 1, and properly equivalent if this can be achieved with determinant 1. The discriminant of the form

$$ax^3 + bx^2y + cxy^2 + dy^3 \tag{1}$$

is the invariant

$$D = 18abcd + b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2, \tag{2}$$

and this has the same value for equivalent forms. The forms of given discriminant, if there are any, fall into a finite number of classes of equivalent forms, or alternatively, of properly equivalent forms. We shall restrict ourselves to those classes which consist of irreducible forms, that is, forms which cannot be expressed as the product of a linear form and a quadratic form with rational coefficients. The object of this paper is to prove the following result.

THEOREM. *If $h(D)$ denotes the number of classes of properly equivalent irreducible forms of discriminant D, then*

$$\sum_{D=1}^{X} h(D) = \frac{\pi^2}{108} X + O(X^{15/16}) \tag{3}$$

as $X \to \infty$.

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† For references, see Dickson's *History of the theory of numbers*, vol. 3, chapter 12.