The sum of the expressions (2) and (3) is an upper estimate for the left-hand side of (1). Adding these two expressions together, we obtain from (3) the term h^n when $m=0$, and for $1 \leqslant m \leqslant n-1$ we obtain

$$\sum_{i_1<\ldots<i_m} \int dx_{i_1} \ldots \int f(x_{i_1}, \ldots, x_{i_m})\, dx_{i_m}.$$

This gives the right-hand side of (1), and so proves the result.

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ON THE CLASS-NUMBER OF BINARY CUBIC FORMS (I)

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1. The arithmetical theory of binary cubic forms with integral coefficients was founded by Eisenstein, and further contributions were made by Arndt, Hermite and others†. Two such forms are said to be equivalent if one can be transformed into the other by a linear substitution with integral coefficients and determinant ± 1, and properly equivalent if this can be achieved with determinant 1. The discriminant of the form

$$ax^3+bx^2y+cxy^2+dy^3 \tag{1}$$

is the invariant

$$D = 18abcd+b^2c^2-4ac^3-4b^3d-27a^2d^2, \tag{2}$$

and this has the same value for equivalent forms. The forms of given discriminant, if there are any, fall into a finite number of classes of equivalent forms, or alternatively, of properly equivalent forms. We shall restrict ourselves to those classes which consist of irreducible forms, that is, forms which cannot be expressed as the product of a linear form and a quadratic form with rational coefficients. The object of this paper is to prove the following result.

THEOREM. *If $h(D)$ denotes the number of classes of properly equivalent irreducible forms of discriminant D, then*

$$\sum_{D=1}^{X} h(D) = \frac{\pi^2}{108} X + O(X^{\frac{15}{16}}) \tag{3}$$

as $X \to \infty$.

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* Received and read 15 June, 1950.

† For references, see Dickson's *History of the theory of numbers*, vol. 3, chapter 12.

The corresponding result for primitive forms (that is, forms for which a, b, c, d have highest common factor 1) follows from this in the usual way, and differs only in that the constant on the right of (3) has then to be divided by $\zeta(4)$.

2. The proof is based on Hermite's definition of a reduced form. The binary cubic form (1) has the quadratic covariant

$$Ax^2+Bxy+Cy^2, \tag{4}$$

where

$$A = b^2-3ac, \; B = bc-9ad, \; C = c^2-3bd; \tag{5}$$

this quadratic form being the Hessian of the cubic form, apart from a numerical factor. If $D > 0$, the quadratic covariant is positive definite, and its discriminant has the absolute value

$$4AC-B^2 = 3D. \tag{6}$$

By the classical theory of binary quadratic forms, any cubic form is properly equivalent to one whose quadratic covariant is reduced, that is, satisfies

$$\begin{cases} \text{either} & -A < B \leqslant A < C \\ \text{or} & 0 \leqslant B \leqslant A = C. \end{cases} \tag{7}$$

Such a cubic is said to be *reduced*.

If two reduced cubics are properly equivalent, their quadratic covariants must be properly equivalent, and so must be identical. Moreover, it is known that the only substitutions which transform a reduced quadratic form into itself are $x = x'$, $y = y'$ and $x = -x'$, $y = -y'$, apart from exceptions when the quadratic form is proportional to x^2+y^2 or x^2+xy+y^2. In these cases, there are two or four other substitutions with the property. Hence two reduced cubics which are equivalent must be either identical or identically opposite, apart from these possible exceptions.

Since $a \neq 0$ for irreducible forms, it follows that every class of properly equivalent irreducible cubic forms is represented by exactly one reduced form with $a > 0$, apart from possible exceptions when $A = C$ and $B = 0$ or $A = B = C$. In these cases there may, as far as we know, be one, two or three reduced forms which all belong to the same class. If $h'(D)$ denotes the number of irreducible reduced* forms with $a > 0$, and $h_1'(D)$ denotes the same number, but excluding any forms for which $A = C$ and $B = 0$, or $A = B = C$, we have

$$h_1'(D) \leqslant h(D) \leqslant h'(D). \tag{8}$$

* The terminology is unfortunate, but can hardly be avoided.

3. We first establish some inequalities for the coefficients of a reduced form with $1 \leqslant D \leqslant X$, or rather, a form which satisfies the simpler conditions (9). These inequalities do not depend upon a, b, c, d being integers.

LEMMA 1. *Let a, b, c, d be real numbers, and let A, B, C be the functions defined in* (5). *Suppose that*

$$|B| \leqslant A \leqslant C \quad \textit{and} \quad 0 < D \leqslant X. \tag{9}$$

Then

$$|a| < X^{\frac{1}{4}}, \quad |b| < 2X^{\frac{1}{4}}, \tag{10}$$

$$|ad| < X^{\frac{1}{2}}, \quad |bc| < 4X^{\frac{1}{2}}, \tag{11}$$

$$|ac^3| < 8X, \quad |b^3 d| < 8X, \tag{12}$$

$$c^2 |bc - 9ad| < 4X. \tag{13}$$

Proof. From (5), we have the identities

$$9Ca^2 - 3Bab + Ab^2 = A^2,$$

$$Cc^2 - 3Bcd + 9Ad^2 = C^2.$$

In each of these equations, the central term on the left does not exceed, in absolute value, the geometrical mean of the two other terms, since $B^2 \leqslant AC$. Hence it does not exceed half their sum, and we have

$$9Ca^2 + Ab^2 \leqslant 2A^2,$$

$$Cc^2 + 9Ad^2 \leqslant 2C^2.$$

Thus

$$|a| < AC^{-\frac{1}{2}}, \quad |b| < 2A^{\frac{1}{2}}, \quad |c| < 2C^{\frac{1}{2}}, \quad |d| < CA^{-\frac{1}{2}}. \tag{14}$$

Since $A \leqslant C$ and

$$AC \leqslant \tfrac{1}{3}(4AC - B^2) = D \leqslant X$$

by (9), the results (10), (11), (12) follow.

Also, by (5) and (9),

$$|bc - 9ad| = |B| \leqslant A.$$

Combining this with the third of the inequalities (14), we obtain (13). This proves the lemma.

4. The preceding lemma allows us to effect some slight simplifications.

LEMMA 2. *The number of cubic forms, with integral coefficients and $a > 0$, which satisfy* (9) *and for which $|B| = A$ or $A = C$, is*

$$O(X^{\frac{3}{4}} \log X).$$

Proof. If $|B| = A$, we have

$$bc - 9ad = \pm(b^2 - 3ac),$$

and so d is determined by a, b, c with at most two possibilities. The number of choices for b is $O(X^{1/4})$ by (10), and the number of choices for a and c is

$$O\Big(\sum_a X^{1/3} a^{-1/3}\Big) = O\big(X^{1/3}(X^{1/4})^{2/3}\big) = O(X^{1/2}),$$

by (12) and (10).

If $A = C$, we have

$$b^2 - 3ac = c^2 - 3bd,$$

and so c is determined by a, b, d, with at most two possibilities. Again $b = O(X^{1/4})$, and the number of choices for a and d is

$$O\Big(\sum_a X^{1/2} a^{-1}\Big) = O(X^{1/2} \log X).$$

This completes the proof of Lemma 2.

It follows from (8) and Lemma 2 that

$$\sum_{D=1}^{X} h(D) = \sum_{D=1}^{X} h'(D) + O(X^{3/4} \log X). \tag{15}$$

It follows also from Lemma 2 that this remains true if, in the definition of $h'(D)$, we understand by a reduced cubic simply one which satisfies

$$|B| \leqslant A \leqslant C. \tag{16}$$

We shall therefore take $h'(D)$, henceforth, to be the number of irreducible cubics with $a > 0$, of discriminant D, which satisfy (16).

LEMMA 3. *The number of reducible cubic forms with $a > 0$ which satisfy* (16), *and for which $0 < D \leqslant X$, is $O(X^{3/4+\epsilon})$ as $X \to \infty$, for any $\epsilon > 0$.*

Proof. Consider first forms for which $d = 0$. By (10) and (12) of Lemma 1, the number of choices for a, b, c is

$$O\Big(X^{1/4} \sum_a X^{1/3} a^{-1/3}\Big) = O(X^{3/4}).$$

Now consider forms for which $d \neq 0$. For a reducible form, there exist relatively prime integers r, s such that

$$ar^3 + br^2 s + crs^2 + ds^3 = 0.$$

Plainly r is a factor of d, and s is a factor of a. Writing $a = a_1 s$ and $d = d_1 r$, we have

$$a_1 r^2 + br + cs + d_1 s^2 = 0.$$

It follows that c is uniquely determined by a_1, d_1, r, s, b. Since $a_1 d_1 rs$ is a non-zero integer, numerically less than $X^{\frac{1}{2}}$, the number of choices for a_1, d_1, r, s is $O(X^{\frac{1}{2}+\epsilon})$ for any $\epsilon > 0$. Also $b = O(X^{\frac{1}{4}})$, whence the result.

We now define $h''(D)$ to be the number of cubic forms with $a > 0$, of discriminant D, which satisfy (16), and have

$$\sum_{D=1}^{X} h(D) = \sum_{D=1}^{X} h''(D) + O(X^{\frac{3}{4}+\epsilon}). \tag{17}$$

5. Having thus simplified the problem, our next step is to complicate it again. We write $\eta = \frac{1}{16}$.

LEMMA 4. *The number of cubic forms with $a > 0$ which satisfy* (16) *and $0 < D \leqslant X$, and for which $a < X^{\eta}$, is $O(X^{\frac{11}{12}+\frac{1}{3}\eta})$.*

Proof. When a, b, c are fixed, the value of d is restricted by

$$a|d| < X^{\frac{1}{2}}, \quad |b^3 d| < 8X, \quad c^2|bc-9ad| < 4X.$$

Hence the number of possibilities for d is*

$$O\{\min(X^{\frac{1}{2}}a^{-1}, \quad X|b|^{-3}, \quad Xa^{-1}c^{-2})\}. \tag{18}$$

It suffices to sum this over a, b, c subject to $0 < a < X^{\eta}$. Write

$$m = m(a, b) = \min(X^{\frac{1}{2}}a^{-1}, \ X|b|^{-3}).$$

Summation of (18) over c gives

$$O\Big(\sum_{c} \min(m, Xa^{-1}c^{-2})\Big) = O(mX^{\frac{1}{2}}a^{-\frac{1}{2}}m^{-\frac{1}{2}}),$$

on noting that $X^{\frac{1}{2}}a^{-\frac{1}{2}}m^{-\frac{1}{2}} > 1$, since $m \leqslant X^{\frac{1}{2}}a^{-1}$. Summation over b now gives

$$O\Big\{X^{\frac{1}{2}}a^{-\frac{1}{2}}\sum_{b}\min(X^{\frac{1}{4}}a^{-\frac{1}{2}}, \ X^{\frac{1}{2}}|b|^{-\frac{3}{2}})\Big\} = O(X^{\frac{1}{2}}a^{-\frac{1}{2}}X^{\frac{1}{4}}a^{-\frac{1}{2}}X^{\frac{1}{6}}a^{\frac{1}{3}}) = O(X^{\frac{11}{12}}a^{-\frac{2}{3}}).$$

Finally, summing this over a, we obtain $O(X^{\frac{11}{12}+\frac{1}{3}\eta})$.

6. By (17) and Lemma 4, the proof of the theorem will be complete, apart from the evaluation of the constant, when the following lemma has been established. Note that

$$\tfrac{11}{12}+\tfrac{1}{3}\eta = 1-\eta = \tfrac{15}{16}.$$

* If b or c is zero, the corresponding term is to be omitted.

LEMMA 5. *The number of sets of integers a, b, c, d which satisfy* (16) *and* $0 < D \leqslant X$ *and*

$$a \geqslant X^{\eta}, \tag{19}$$

is

$$KX + O(X^{1-\eta}) \tag{20}$$

as $X \to \infty$, *where* K *is a positive absolute constant.*

Proof. Let $\mathcal{R}$ denote the region in four dimensional space consisting of all points $(\alpha, \beta, \gamma, \delta)$ which satisfy

$$|\beta\gamma - 9\alpha\delta| \leqslant \beta^2 - 3\alpha\gamma \leqslant \gamma^2 - 3\beta\delta, \tag{21}$$

$$4(\beta^2 - 3\alpha\gamma)(\gamma^2 - 3\beta\delta) - (\beta\gamma - 9\alpha\delta)^2 \leqslant X, \tag{22}$$

$$\alpha \geqslant X^{\eta}. \tag{23}$$

The number we have to investigate is the number N of points with integral coordinates in $\mathcal{R}$. By Lemma 1, the inequalities (10), (11), (12), with $\alpha, \beta, \gamma, \delta$ in place of a, b, c, d, are valid for all points of $\mathcal{R}$. Hence $\mathcal{R}$ is bounded, since $|\gamma|^3 < 8X^{1-\eta}$ and $|\delta| < X^{\frac{1}{2}-\eta}$, on using (23).

Since the boundary of $\mathcal{R}$ consists of a bounded number of portions of algebraic surfaces of bounded degrees, the region satisfies the conditions of a theorem recently established elsewhere* Let V denote the volume of $\mathcal{R}$. Let V_{α} denote the volume of the projection of $\mathcal{R}$ on the space $\alpha = 0$, and $V_{\alpha\beta}$ the area of the projection of $\mathcal{R}$ on the plane $\alpha = \beta = 0$, and $V_{\alpha\beta\gamma}$ the length of the projection of $\mathcal{R}$ on the line $\alpha = \beta = \gamma = 0$. By the theorem referred to,

$$N - V = O\{\max(V_{\alpha}, \ldots, V_{\alpha\beta}, \ldots, V_{\alpha\beta\gamma}, \ldots, 1)\}, \tag{24}$$

where the constant implied by the symbol O is an absolute constant.

We first estimate V_{α}. For a point in the projection of $\mathcal{R}$ on $\alpha = 0$, we have

$$|\gamma| < \min(2X^{\frac{1}{3}(1-\eta)},\ 4X^{\frac{1}{2}}|\beta|^{-1}),$$

$$|\delta| < \min(X^{\frac{1}{2}-\eta},\ 8X|\beta|^{-3}).$$

Writing

$$m = m(\beta) = \min(2X^{\frac{1}{3}(1-\eta)},\ 4X^{\frac{1}{2}}|\beta|^{-1}),$$

* "Note on a principle of Lipschitz", this *Journal*, 26 (1951), 179–183.

we have $|\delta| < X^{-\frac12} m^3$, and consequently*

$$V_\alpha = O\left(\int_0^\infty d\beta \int_0^m X^{-\frac12} m^3\, d\gamma\right)$$

$$= O\left(\int_0^\infty X^{-\frac12} m^4\, d\beta\right)$$

$$= O\left(X^{-\frac12} \int_0^\infty \min(X^{\frac43(1-\eta)},\ X^2 \beta^{-4})\, d\beta\right)$$

$$= O(X^{-\frac12} X^{\frac43(1-\eta)} X^{\frac12-\frac13(1-\eta)}) = O(X^{1-\eta}).$$

The estimations of V_β, V_γ, V_δ are simpler. For a point in the projection of $\mathcal{R}$ on $\beta = 0$, we have

$$\alpha \geqslant X^\eta, \quad |\gamma| < 2X^{\frac13}\alpha^{-\frac13}, \quad |\delta| < X^{\frac12}\alpha^{-1},$$

whence

$$V_\beta = O\left(\int_{X^\eta}^\infty X^{\frac13}\alpha^{-\frac13} X^{\frac12}\alpha^{-1}\, d\alpha\right) = O(X^{\frac56}).$$

In the projection of $\mathcal{R}$ on $\gamma = 0$, we have

$$0 < \alpha < X^{\frac14}, \quad |\delta| < \min(X^{\frac12}\alpha^{-1},\ 8X|\beta|^{-3}),$$

whence

$$V_\gamma = O\left(\int_0^{X^{\frac14}} d\alpha \int_0^\infty \min(X^{\frac12}\alpha^{-1},\ X\beta^{-3})\, d\beta\right) = O(X^{\frac34}).$$

In the projection of $\mathcal{R}$ on $\delta = 0$, we have

$$0 < \alpha < X^{\frac14}, \quad |\beta| < 2X^{\frac14}, \quad |\gamma| < 2X^{\frac13(1-\eta)},$$

whence

$$V_\delta = O(X^{\frac56}).$$

For the remaining projections, a very crude estimate suffices. We have

$$0 < \alpha < X^{\frac14},\ |\beta| < 2X^{\frac14},\ |\gamma| < 2X^{\frac13},\ |\delta| < X^{\frac12};$$

and so all two dimensional and one dimensional projections of $\mathcal{R}$ have area or length $O(X^{\frac56})$.

It follows now from (24) that

$$N - V = O(X^{1-\eta}). \tag{25}$$

We denote by V' the volume of the four dimensional region $\mathcal{R}'$ defined by (21) and (22), with (23) replaced by $\alpha \geqslant 0$. It should be observed that the region $\mathcal{R}'$ is not bounded, since it contains, for example, the point

* It is for this estimate that (19) is needed.

$(\xi^{-3},\ -3\xi^{-1},\ -3\xi,\ \xi^3)$, where ξ is arbitrarily large. We proceed to prove that V' is nevertheless finite. The proof follows similar lines to that of Lemma 4. When α, β, γ have given non-zero values, the value of δ is restricted to intervals whose total length is

$$O\Big(\min(X^{\frac{1}{2}}\alpha^{-1},\ X|\beta|^{-3},\ X\alpha^{-1}\gamma^{-2})\Big).$$

Integrating this over γ from $-\infty$ to ∞, we obtain

$$O(X^{\frac{1}{2}}\alpha^{-\frac{1}{2}}m^{\frac{1}{2}}),$$

as in the proof of Lemma 4, where

$$m = m(\alpha, \beta) = \min(X^{\frac{1}{2}}\alpha^{-1},\ X|\beta|^{-3}).$$

Integration over β from $-\infty$ to ∞ gives $O(X^{\frac{11}{12}}\alpha^{-\frac{2}{3}})$ Integration over α from 0 to $X^{\frac{1}{4}}$ gives $O(X)$. Hence V' is finite.

The same calculation allows us to estimate $V'-V$. We have only to modify the last step by integrating over α from 0 to X^{η}, obtaining $O(X^{\frac{11}{12}+\frac{1}{3}\eta})$. Hence

$$V'-V = O(X^{\frac{11}{12}+\frac{1}{3}\eta}),$$

and it follows from (25) that

$$N = V'+O(X^{\frac{15}{16}}). \tag{26}$$

The volume V' was defined by the inequalities (21), (22), and $\alpha \geqslant 0$. By considerations of homogeneity, $V' = KX$, where K is a positive absolute constant, namely the volume of the four dimensional region defined by $\alpha > 0$ and

$$|\beta\gamma-9\alpha\delta| < \beta^2-3\alpha\gamma < \gamma^2-3\beta\delta, \tag{27}$$

$$4(\beta^2-3\alpha\gamma)(\gamma^2-3\beta\delta)-(\beta\gamma-9\alpha\delta)^2 < 1. \tag{28}$$

Substituting in (26), we obtain (20).

7. The evaluation of K is best carried out in stages. The inequalities (27) and (28) can be simplified by replacing α, δ by $\frac{1}{3}\alpha$, $\frac{1}{3}\delta$. Observing that the sign of B can be changed without disturbing A or C by changing the signs of β and δ, we see that K is $\frac{2}{9}$ of the volume of the region defined by

$$\alpha > 0, \quad 0 < \beta\gamma-\alpha\delta < \beta^2-\alpha\gamma < \gamma^2-\beta\delta,$$

$$4(\beta^2-\alpha\gamma)(\gamma^2-\beta\delta)-(\beta\gamma-\alpha\delta)^2 < 1.$$

We write

$$P = \beta^2-\alpha\gamma, \quad Q = \beta\gamma-\alpha\delta, \quad R = \gamma^2-\beta\delta.$$

There is a one-to-one correspondence between δ and R when β and γ are fixed, and $d\delta/dR = -\beta^{-1}$. There is also a one-to-one correspondence between γ and P when α and β are fixed, and $d\gamma/dP = -\alpha^{-1}$. Hence

$$K = \tfrac{2}{9}\iiiint \alpha^{-1}|\beta|^{-1}\,d\alpha\,d\beta\,dP\,dR,$$

extended over the region

(29) $$\alpha > 0,\quad 0 < Q < P < R,\quad 4PR - Q^2 < 1.$$

Here Q must be regarded as a function of α, β, P, R; and its expression in this form is easily seen to be

$$Q = P\beta\alpha^{-1} + R\alpha\beta^{-1} - P^2\alpha^{-1}\beta^{-1}.$$

We divide K into K_1 and K_2, corresponding to $\beta > 0$ and $\beta < 0$. In the former case, we put

(30) $$\alpha = \xi^{-\frac{1}{2}}\eta^{-\frac{1}{2}},\quad \beta = \xi^{-\frac{1}{2}}\eta^{\frac{1}{2}},$$

obtaining

$$K_1 = \tfrac{1}{9}\iiiint \xi^{-1}\eta^{-1}\,d\xi\,d\eta\,dP\,dR,$$

extended over positive ξ, η, P, R satisfying (29), where now

$$Q = P\eta + R\eta^{-1} - P^2\xi.$$

To simplify the conditions further, we transform from ξ, η, P, R to T, θ, u, v, where

$$\xi = T^{-1}\theta^{-2}u,\quad \eta = \theta^{-1}v,\quad P = T\theta,\quad R = T\theta^{-1}.$$

We obtain

$$K_1 = \tfrac{2}{9}\iiiint T\theta^{-1}u^{-1}v^{-1}\,dT\,d\theta\,du\,dv,$$

extended over positive T, θ, u, v satisfying

$$0 < v + v^{-1} - u < \theta < 1,$$

$$4T^2 - T^2(v + v^{-1} - u)^2 < 1.$$

Writing $u = v + v^{-1} - w$, we have

$$K_1 = \tfrac{2}{9}\iiiint \frac{T\,dT\,d\theta\,dv\,dw}{\theta(v^2 + 1 - vw)},$$

extended over

(31) $$0 < w < \theta < 1,\quad T^2 < (4 - w^2)^{-1},\quad T > 0,\quad v > 0.$$

The condition arising from $u > 0$ is necessarily satisfied, since $w < 1$ and $v + v^{-1} \geqslant 2$.

A similar treatment applies to K_2, on writing $\beta = -\xi^{-\frac{1}{3}}\eta^{\frac{1}{3}}$ in (30). The final formula differs only in the sign of w in the denominator. By addition of the two formulae,

$$K = \tfrac{4}{9}\int\int\int\int \frac{T(v^2+1)\,dT\,d\theta\,dv\,dw}{\theta\{(v^2+1)^2-v^2w^2\}},$$

extended over (31). Integrating over v from 0 to ∞, we have

$$\int_0^\infty \frac{(v^2+1)\,dv}{(v^2+1)^2-v^2w^2} = \int_{-\infty}^{\infty} \frac{dx}{x^2+4-w^2} = \frac{\pi}{\sqrt{(4-w^2)}}.$$

by the substitution $v-v^{-1}=x$. Also, integrating over T, we have

$$\int_0^{1/\sqrt{(4-w^2)}} T\,dT = \tfrac{1}{2}(4-w^2)^{-1}.$$

Hence

$$K = \frac{2\pi}{9}\int_0^1 \frac{d\theta}{\theta}\int_0^\theta \frac{dw}{(4-w^2)^{\frac{3}{2}}}.$$

Writing $\theta = 2\sin\phi$ and $w = 2\sin\psi$, we obtain

$$K = \frac{2\pi}{9}\int_0^{\pi/6} \cot\phi\,d\phi \int_0^\phi \tfrac{1}{4}\sec^2\psi\,d\psi = \frac{\pi^2}{108}.$$

This gives (3).

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ON THE CLASS-NUMBER OF BINARY CUBIC FORMS (II)

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1. In the preceding paper (which will be referred to as I), I obtained an asymptotic formula for $\sum_{D=1}^{X} h(D)$, where $h(D)$ denotes the number of classes of properly equivalent irreducible binary cubic forms, with integral coefficients, of given discriminant D. The object of the present note is to prove a similar result for forms of negative discriminant, which is as follows.

THEOREM. *We have*

$$\sum_{\Delta=1}^{X} h(-\Delta) = \frac{\pi^2}{12}X + O(X^{\frac{15}{16}}) \tag{1}$$

as $X \to \infty$.

* Received and read 15 June, 1950.