Exercise Set 7 – Arithmetic Geometry, Frank Thorne (thorne@math.sc.edu)

Due Wednesday, April 13, 2016

(1) Verify that the map ϕ from page 28.5 of the lecture notes does indeed map E to $\overline{E}$.

(2) Suppose that P_1, P_2, and P_3 are collinear points on E. Prove that ϕ maps them to collinear points on $\overline{E}$, and explain why this means that ϕ is a homomorphism.

You are welcome to pass to a special case which includes 'most' possibilities. In particular you may assume that these points are distinct and none is the point at infinity. Make additional assumptions if they are helpful, but be explicit about what you are assuming.

(3) (**Important. If you skip the other two, fine, but at least do this one!**) Recall that for an elliptic curve of the form $y^2 = x^3 + ax^2 + bx$, we have a homomorphism α from $E(\mathbb{Q})$ to $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ as defined on p. 29.3.

Let E be the curve $y^2 = x(x-2)(x-10)$.

(a) Compute the associated Selmer group of E – i.e., the finite subgroup of $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ described on p. 29.3 in which the image of α is guaranteed to lie.

(b) Compute the image of α. (A good way to get started is to write down a bunch of rational points on E and see where they map to. Remember also that α is a homomorphism, so that if you know where P_1 and P_2 go, you also know where $P_1 + P_2$ goes to.)

This will (if I'm not mistaken) not be the entire Selmer group. Can you prove this?