Exercise Set 6 – Arithmetic Geometry, Frank Thorne (thorne@math.sc.edu)

Due Monday, March 28, 2016

(1) A special case of **Schanuel's Theorem** says that

$$C(\mathbb{P}^N(\mathbb{Q}), B) := \{P \in \mathbb{P}^N(\mathbb{Q}) \ : \ H(P) \leq B\} \sim \frac{2^N}{\zeta(N+1)} B^{N+1}.$$

The following exercise outlines a proof of this theorem.

(a) Prove that the set to be counted is in an exactly 2-to-1 correspondence with $N+1$-tuples of integers $(x_0, x_1, \cdots, x_N)$, where $x_i \in [-B, B]$ for each i, and the x_i do not all share a common factor.

(b) Write $C(N, B)$ for the number of $N+1$-tuples of integers $(x_0, x_1, \cdots, x_N)$, where $x_i \in [-B, B]$ for each i (but with no 'no common factor' condition). Prove that

$$C(N, B) = (2B)^{N+1} + O(B^N).$$

(Be sure to justify that the constant implied by the O-notation is in fact independent of B. It may depend on N however. You might wish to be especially careful and actually compute the constant implied in the error term.)

(c) Let $\mu(d)$ be the *Möbius function*, equal to $(-1)^{\omega(d)}$ if d is squarefree, where $\omega(d)$ denotes the number of prime factors of d, and equal to zero otherwise.

Prove that, for any positive integer n, $\sum_{d|n} \mu(d)$ is equal to 1 if $n = 1$ and zero otherwise. (Hint: the sum can be rewritten as $\prod_{p|n}(1 + \mu(p))$, where the product is over all primes dividing n. Why is this?)

(d) Write $C(N, B, d)$ for the number of $N+1$-tuples of integers $(x_0, x_1, \cdots, x_N)$, where $x_i \in [-B, B]$ for each i, such that d divides all the x_i. Explain why $C(N, B, d) = C(N, B/d)$ and deduce an estimate for $C(N, B, d)$.

(e) Prove that

$$C(\mathbb{P}^N(\mathbb{Q}), B) = \sum_{d=1}^{B} \mu(d) C(N, B, d)$$

and use your previous estimates to conclude that

$$C(\mathbb{P}^N(\mathbb{Q}), B) = 2^N B^{N+1} \sum_{d=1}^{B} \frac{\mu(d)}{d^{B+1}} + o(B^{N+1}).$$

(f) Prove that

$$\sum_{d=B+1}^{\infty} \frac{\mu(d)}{d^{N+1}} = o(1)$$

and that

$$\sum_{d=1}^{\infty} \frac{\mu(d)}{d^{N+1}} = \frac{1}{\zeta(B+1)}.$$

Conclude that
$$\sum_{d=1}^{B} \frac{\mu(d)}{d^{N+1}} = \frac{1}{\zeta(N+1)} + o(1).$$

(g) Conclude the statement of Schanuel's theorem.

Note that Schanuel proved his result where $\mathbb{Q}$ is replaced with any number field, where the analysis became more difficult. For definitions of height functions in number fields, see Chapter 8.5 of Silverman.