Exercise Set 4 – Arithmetic Geometry, Frank Thorne (thorne@math.sc.edu)

Due Friday, February 19, 2016

This homework presumes knowledge of basic complex analysis: meromorphic functions and Laurent series expansions in a neighborhood of any point; contour integrals; residues; Cauchy's residue theorem. If you do not know these topics, please put this homework off until you do.

(1) For a given lattice, prove the Laurent series expansion (around $z = 0$)

$$\wp(z) = z^{-2} + 3G_4 z^2 + 5G_6 z^4 + 7G_8 z^6 + O(z^8)$$

Now, compute the Laurent series expansions for $\wp'(z)$ (up to $O(z^8)$) and $\wp(z)^3$ and $\wp'(z)^2$ (up to $O(z^2)$). Conclude that

$$\wp'(z)^2 - \Big(4\wp(z)^3 - g_2\wp(z) - g_3\Big) = O(z^2)$$

in a neighborhood of zero, and then explain why this forces this function to be identically zero.

(2) (a) Suppose that $f(z) = (z - z_0)^k g(z)$, where $g(z)$ is holomorphic in a neighborhood of $z = z_0$. Compute the residues of $\frac{f'(z)}{f(z)}$ and $m\frac{f'(z)}{f(z)}$ at $z = z_0$.

(b) Let $f(z)$ be an elliptic function with respect to the lattice Λ, and let D be a fundamental parallelogram chosen so that its boundary ∂D does not pass through any zeroes or poles of D. (Why can such a choice be made?)
Prove that

$$\sum_{z \in D} \mathrm{ord}_z(f) = 0,$$

where $\mathrm{ord}_z(f)$ indicates the order of zero (if positive) or pole, by evaluating the integral

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)} dz$$

in two different ways: directly (by cancelling opposite sides), and by using Cauchy's residue theorem.

(c) With the same setup as in the previous problem, let $a_1, \dots, a_n$ be the zeroes of f (counted, as always, with multiplicity), and let $b_1, \cdots, b_n$ be the poles. Prove that

$$\sum a_i - \sum b_j \equiv 0 \pmod{\Lambda},$$

by evaluating the integral

$$\frac{1}{2\pi i}\int_{\partial D} z\frac{f'(z)}{f(z)} dz$$

in two different ways: directly, and by using Cauchy's residue theorem.
Hint: For any meromorphic function $g(z)$ with $g(a) = g(b)$, the integral $\frac{1}{2\pi i}\int_a^b \frac{g'(z)}{g(z)} dz$ is the winding number around 0 of the path

$$[0,1] \to \mathbb{C}, \qquad t \to g((1-t)a + tb),$$

and in particular is an integer.