Math 788E - Elliptic Curves and Arithmetic Geometry

Spring 2020

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Meeting times: MWF 12-12:50

I. Prerequisites

Graduate standing and Math 702 (concurrent enrollment is okay), or permission of instructor.

II. Learning Outcomes

Successful students will:

- Master a variety of techniques for finding rational and integral points on varieties when they do exist, and proving nonexistence when they don't. Students will, at least, be able to describe the set of rational points on conics and on elliptic curves.
- Master the basics of elliptic curves: what are they, what is the group law, how do you find the rational points, and what is the structure of the group of rational points. Be able to describe characteristics of elliptic curves over Q, over C, and over finite fields.
- Witness (and, if they put in significant additional effort, master) classical proofs in the subject, in particular that of the Mordell-Weil theorem.
- Understand the statement of the Birch and Swinnerton-Dyer Conjecture, one of the seven Millenium Problems (and one of the six unsolved Millenium Problems!), worth a bounty of $1,000,000.
- Study the relevance of algebraic geometry for problems in number theory. Students who have studied algebraic geometry will learn how these techniques are applied by number theorists. Students who have not will see its beauty and power, and will hopefully be motivated to study algebraic geometry.
- Gain exposure to a wide variety of problems in arithmetic geometry, as well as the wide variety of mathematical tools (representation theory, harmonic and complex analysis, algebraic number theory, commutative algebra, scheme theory, modular forms, etc., etc., etc.) which go into its study. There is far too much material to master in a lifetime, let alone in a semester. The student will gain an appreciation for the subject in broad outline, sufficient to better understand conference talks and some of the research literature, and opening doors to the study of further special topics.

III. Textbook

The course is intended for students with varying amounts of algebraic background and interest. Much of the course will follow the following two books:

Silverman and Tate, Rational Points on Elliptic Curves (Springer)

Silverman, The Arithmetic of Elliptic Curves (Springer)

It is recommended that students buy one and follow along. To a large extent they cover the same material in different ways. Silverman and Tate is easier. Silverman's book is the better choice for anyone who has, or wants to acquire, a more serious algebraic background.

IV. Instructional Delivery

Lecture.

V. Course Requirements

The course grade will be based on homework assignments entirely, given approximately every week or two. Each will count equally. There will be no exams.

Some of the homework sets may feature a choice – larger numbers of easier, computational problems, or a single more difficult problem. The latter is a good choice for students with (or wishing to acquire) more algebraic background.

VI. Grading

Students are guaranteed at least: A for 80%+, B for 60%+, C for 40%+, D for 30%+. Anything lower is subject to a failing grade.

VII. Course Policies

Attendance

Attendance will not be recorded, and no formal attendance policy will be enforced.

Honor Code

Students are expected to abide by the Honor Code. Collaboration is encouraged on the homeworks but you should write up your own solutions.

Disabilities

Any student with a documented disability should contact the Office of Student Disability Services on the first floor of LeConte to make arrangements for appropriate accommodations.

VIII. Course Outline

The following is a rough outline of topics that are likely to be covered:

- (2 weeks) Introduction, the geometry of conics, Bezout's theorem.
- (4 weeks) Elliptic curves over C: the group law, divisors and the Picard group, torsion, CM, the j-invariant.
- (2 weeks) Elliptic curves over finite fields: the Weil conjectures, Hasse-Weil zeta functions.
- (3 weeks) Elliptic curves over Q; height functions, the Mordell-Weil theorem.
- (3 weeks) Torsors, the Selmer and Shafarevich-Tate groups, and the Birch and Swinnerton-Dyer conjecture.