Exercise Set 2 – Arithmetic Geometry, Frank Thorne (thorne@math.sc.edu)

Due Monday, February 10, 2016

Instructions. Do either 1-3 or 4.

(1) Let E be the elliptic curve $y^2 = x^3 + 17$, and let $P = (-2, 3)$ and $Q = (2, 5)$.
Compute $-P + 2Q$ and $3P + Q$.

(2) (a) Explain why the following statement is true: Let P be a point on an elliptic curve E. Then, P is a 3-torsion point if and only if the tangent line at E through P does not meet the curve at any other point.

(b) If E is the curve $y^2 = x^3 + Ax + B$, what is the tangent line to the curve at the point at infinity? Compute it (possibly in projective coordinates, or in a different affine patch) and verify that it doesn't intersect E at any other point.

(c) If E is $y^2 = x^3 + 1$, use this criterion to show that $(0, \pm 1)$ are 3-torsion points.

(d) Let $E : y^2 = x^3 + Ax + B$ be an elliptic curve, and suppose that $(x_0, y_0) \in E$.
Compute the tangent line to E at (x_0, y_0), and substitute it into E to obtain a formula for the x-coordinate of the third intersection point of E with this tangent line. Deduce an algebraic formula necessary for (x_0, y_0) to be a 3-torsion point.

(3) Again let E be the elliptic curve given by the equation

$$y^2 = x^3 + Ax + B$$

for some complex numbers A and B. Prove that the following conditions are all equivalent.

(i) E is smooth. (Recall that you must homogenize the curve, write E as $V(f)$, and check that $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$ do not simultaneously vanish on any point of E. (They are allowed to vanish on points not on E.)

(ii) The cubic $x^3 + Ax + B$ has distinct roots.

(iii) The discriminant $\Delta = -16(4A^3 + 27B^2)$ is nonzero.

(4) (Presentation taken from Garrity et al., *Algebraic Geometry: A Problem Solving Approach.*)

(a) Consider eight distinct points P_1 through P_8 in $\mathbb{P}^2$, so that no four are collinear and no seven are on any conic. Let F be a generic cubic polynomial with unknown coefficients a_1 through a_{10}. The system of simultaneous equations

$$F(P_1) = F(P_2) = \cdots = F(P_8) = 0$$

is a system of eight linear equations in these ten unknowns. Prove that the vector space of solutions to these equations has dimension 2 by considering each of the following cases.

(i) The eight points are in *general position*, which means that no three are collinear and no six are on a conic.

(ii) Three of the points are collinear.

(iii) Six of the points are on a conic.

(b) Show that there are two linearly independent cubics F_1 and F_2, so that any of cubic passing through the eight points P_1 through P_8 has the form $\lambda F_1 + \mu F_2$. Conclude that for any collection of eight points with no four collinear and no seven on a conic, there is a *unique* ninth point P_9 such that *every* cubic curve passing through the eight given points must also pass through P_9.

(c) Conclude the statement of the Cayley-Bacharach theorem given in class and in the lecture notes.