Homework 8, Math 702 – Frank Thorne (thorne@math.sc.edu)

You are welcome and encouraged to collaborate, but please write up your own solutions. **Please appeal to the universal property in your proofs whenever possible.**

Due Friday, February 9, 2018.

1. Compute or describe, as explicitly as you can, the following tensor products.

   (a) $\mathbb{Z}/10\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$.

   (b) (Generalizing the above) $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$, for integers m and n.

   (c) $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$, where: A is a finite abelian group, k is the largest integer such that p^k divides the order of A.

   (d) $S \otimes_R (R[x_1, \ldots, x_n]/I)$, where: R and S are commutative rings with $R \subseteq S$, and I is any ideal in the polynomial ring $R[x_1, \ldots, x_n]$.

2. Let $V = \mathbb{R}^2$. Which elements of $V \otimes_{\mathbb{R}} V$ can and cannot be written as simple tensors (i.e. in the form $v \otimes w$)?

3. Write your own proof of one of the following: the associative law (Theorem 14 in DF), the distributive law (Theorem 17), or the commutative law (Theorem 20) for tensor products.

   Please do so without looking at the corresponding proof in DF or in the class notes. (But reading the other two proofs is encouraged.) **No hack-and-slash**, use the UP.

4. This question will introduce a bit of *representation theory.*

   The group $\mathrm{GL}_2(\mathbb{C})$ acts on the vector space $\mathbb{C}^2$ in the usual way, and it acts on $T^2(\mathbb{C}^2) = \mathbb{C}^2 \otimes \mathbb{C}^2$ by
   $$g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w).$$

   (a) Prove that this action also defines well-defined actions on the quotients $S^2(\mathbb{C}^2)$ and $\wedge^2(\mathbb{C}^2)$. We therefore have actions of $\mathrm{GL}_2(\mathbb{C})$ on $4-$, $3-$, and 1-dimensional vector spaces.

   (b) Explain how these actions yield homomorphisms into $\mathrm{GL}_4(\mathbb{C})$, $\mathrm{GL}_3(\mathbb{C})$, and $\mathrm{GL}_1(\mathbb{C})$ respectively. These are called *representations* of $\mathrm{GL}_2(\mathbb{C})$.

   (c) Choosing suitable bases for all of these vector spaces, compute these homomorphisms explicitly, in terms of matrices.

   (d) A representation $\rho : G \to \mathrm{GL}(V)$ (where G is any group and V is any vector space) is called *irreducible* if there is no invariant proper subspace of V, which is invariant under the action of G.

   Determine which of the three representations above are irreducible.