Homework 6, Math 701 – Frank Thorne (thorne@math.sc.edu)

Instructions: You are welcome and encouraged to collaborate, but please write up your own solutions.

Due Monday, November 20, 2017.

1. Let F be a field. Prove that the matrix ring $M_n(F)$ is *simple*: it has no two-sided ideals other than 0 and itself.

2. Prove (with an explicit description) that $M_2(\mathbb{R})$ contains a subring isomorphic to $\mathbb{C}$.

3. Let R be a commutative ring. Prove that if P is a prime ideal of R that contains no zero divisors, then R is an integral domain.

4. Write $\mathbb{Z}_{(p)}$ for the localization of $\mathbb{Z}$ at (p); that is, the ring of quotients $\frac{a}{b}$ with $b \notin (p)$.

   Prove that $\mathbb{Z}_{(p)}$ is a *local ring*; that is, it has a unique maximal ideal M. Describe M and the residue field $\mathbb{Z}_{(p)}/M$ explicitly.

5. Use the Chinese Remainder Theorem to prove the existence of a partial fraction decomposition: If $g(x) = \prod_{i=1}^{n}(x - a_i) \in \mathbb{C}[x]$ is a polynomial of degree n with distinct roots, and $f(x)$ is any polynomial of lesser degree, then we may write

   $$\frac{f(x)}{g(x)} = \frac{b_1}{x - a_1} + \cdots + \frac{b_n}{x - a_n}$$

   for complex numbers b_i.

   Briefly describe how to amend the proof if g has repeated roots, or if $\mathbb{C}$ is replaced with $\mathbb{R}$ and g is only known to factor into linear and quadratic terms.

6. Prove:

   (a) The maximal ideals of $\mathbb{C}[x]$ are $(x - a)$ for $a \in \mathbb{C}$;

   (b) The only prime, nonmaximal ideal of $\mathbb{C}[x]$ is (0). (Recall that every maximal ideal is prime.)

7. **Bonus.**

   (a) The maximal ideals of $\mathbb{C}[x, y]$ are $(x - a, y - b)$ for $a, b \in \mathbb{C}$;

   (b) The prime, nonmaximal ideals of $\mathbb{C}[x, y]$ are (0) and (f) where f is irreducible.