Homework 4, Math 701 – Frank Thorne (thorne@math.sc.edu)

Instructions: You are welcome and encouraged to collaborate, but please write up your own solutions.
Due Friday, October 13, 2017.

1. If G and H are groups, then their *direct product* $G \times H$ consists of pairs (g, h) with $g \in G$ and $h \in H$, with group operation $(g_1, h_1)(g_2 h_2) = (g_1 g_2, h_1 h_2)$.

   If M and N are normal subgroups of G with $MN = G$, prove that

   $$G/(M \cap N) \simeq (G/M) \times (G/N).$$

2. Prove, for each $n \geq 3$, that A_n contains a subgroup isomorphic to S_{n-2}.

3. Prove that if $G/Z(G)$ is cyclic then G is abelian.

4. Determine all the conjugacy classes of S_6, together with their sizes and the orders of the stabilizers of any element.

   Then, do the same for A_6.

5. Prove that $\mathrm{Aut}(D_4) \simeq D_4$, and describe an isomorphism explicitly.

6. **Challenge Problem.** I got this problem from Section 1 of `http://math.stanford.edu/~vakil/files/sixjan2308.pdf`, which you are encouraged to read. This paper sketches the construction of an outer automorphism of Sym(6), and this problem asks you to fill in the details.

   (a) The *complete graph on* 5 *vertices* consists of five vertices – say, labeled 1 through 5, and the 10 edges between them.

   Suppose you label each edge green or blue. Prove that the green edges form a *Hamiltonian circuit* (a cycle visiting each vertex exactly once) if and only if the blue ones do. Further prove that there are exactly six ways of so coloring this graph. We will call these six colored graphs *mystic pentagons*.

   (b) Prove that the action of Sym(5) on the five vertices induces an action on the six mystic pentagons, inducing a homomorphism $\phi : \mathrm{Sym}(5) \to \mathrm{Sym}(6)$. Further prove that ϕ is injective.

   (c) Explain why the action of Sym(6) on itself by left multiplication induces an action of Sym(6) on the left cosets of $\phi(\mathrm{Sym}(5))$ in Sym(6), and therefore a homomorphism $\sigma : \mathrm{Sym}(6) \to \mathrm{Sym}(6)$.

   (d) Prove that σ is an automorphism of Sym(6), and that it is not induced by conjugation by any element of Sym(6).