Midterm Examination - Math 701, Frank Thorne (thorne@math.sc.edu)

Due Monday, November 6 in class

Instructions:

- This is a **timed, take-home, closed-book exam.** No notes, books, looking for answers on the internet, etc., etc.
- Please take the exam at any time and place of your choosing. I recommend you find some place quiet with no distractions.
- You have **four hours to complete the exam.** Except in case of unforeseen circumstances, it is expected that these be consecutive.
- **After you've finished, please do not discuss the exam before it's due unless you are sure that nobody who hasn't taken it yet is within earshot.**
- You should bring sufficient blank paper and write your answers on this. Alternatively you may TeX it if you wish but this is not expected.
- If you find any questions ambiguous, or if you're not sure if your answer is acceptable, explicitly describe your interpretation and/or concerns as part of your solution.
- **GOOD LUCK!**

1. Compute the number of automorphisms of $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ and of $\mathbb{Z}/25\mathbb{Z}$.

   Solution. Any automorphism ϕ of $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ is determined by $\phi((1,0))$ and $\phi((0,1))$; conversely, choosing any $a, b, c, d \in \mathbb{Z}/5\mathbb{Z}$, the map defined by $\phi(1,0) = (a,b)$ and $\phi(0,1) = (c,d)$ is a homomorphism.

   For any such ϕ, $\text{Im}(\phi)$ has order dividing 25, so it is 1, 5, or 25. If it equals 25 then ϕ is surjective, and therefore injective as well, so it is an automorphism. Therefore we must count the number of such ϕ.

   If $\phi((1,0)) = (0,0)$ then ϕ will fail to be injective, so assume that $\phi((1,0)) = (a,b) \neq (0,0)$. Then the image of ϕ will include all $\mathbb{Z}/5\mathbb{Z}$-multiples of (a,b); if $\phi((0,1)) = (c,d)$ is one of these multiples, then ϕ will fail to be injective, but if it is not a multiple, then $\text{Im}(\phi))$ will have at least six, and hence 25 elements.

   Therefore, for each of the 24 choices of (a,b), there are 20 possible choices of (c,d). So the total number of automorphisms is $24 \cdot 20 = 480$.

   There are many fewer automorphisms of $\mathbb{Z}/25\mathbb{Z}$. As with before, the automorphisms ϕ all satisfy $\phi(1) = a$ for some $a \in \mathbb{Z}/25\mathbb{Z}$, and are determined by this a.

   If $5 \mid a$, then $\text{Im}(\phi)$ will lie in the proper subgroup of multiples of 5; conversely, if $5 \nmid a$, then ϕ will have trivial kernel. (If ab is divisible by 25 and a is coprime to 5, then by elementary number theory b must be a multiple of 25.)

   So the automorphisms are in bijection with the elements of $\mathbb{Z}/25\mathbb{Z}$ which are not multiples of 5, of which there are 20.

2. Consider the usual action of $\text{GL}_2(\mathbb{C})$ on $\mathbb{C}^2$, given by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x + \beta y \\ \gamma x + \delta y \end{pmatrix}.$$

   (a) Compute explicitly the stabilizer H of this action on $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

   Solution. Upon plugging $x = 1$, $y = 0$ into the equation above we see immediately that $\alpha = 1$ and $\gamma = 0$, and so

$$H = \left\{ \begin{pmatrix} 1 & \beta \\ 0 & \delta \end{pmatrix} : \beta \in \mathbb{C}, \delta \in \mathbb{C}^\times \right\}.$$

   Here δ cannot be 0 (otherwise the matrix will be non-invertible) but is otherwise arbitrary.

   (b) Writing $\mathbb{C}$ for the group of complex numbers (with group operation addition) and $\mathbb{C}^\times$ for the group of nonzero complex numbers (with group operation multiplication), prove that H contains subgroups N and K isomorphic to $\mathbb{C}$ and $\mathbb{C}^\times$, respectively. Moreover, prove that N is normal (using any familiar facts from linear algebra if you like).

   Solution. We may take

$$N = \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} : \beta \in \mathbb{C} \right\},$$

$$K = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} : \delta \in \mathbb{C}^\times \right\}.$$

Isomorphisms to $\mathbb{C}$ and $\mathbb{C}^\times$ are defined by taking the β and δ coordinates respectively, and these are isomorphisms because

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \beta + \beta' \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \delta' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \delta\delta' \end{pmatrix}.$$

(c) Prove that H is isomorphic to a semidirect product $N \rtimes_\phi K$ for a nontrivial homomorphism $\phi : K \to \mathrm{Aut}(N)$, which you should describe. N is normal because it is the kernel of the determinant homomorphism to $\mathbb{C}^\times$.

Solution. This follows because N and K are subgroups of H which intersect in the identity, and for which $NK = H$. To check the latter, observe that

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} 1 & \beta\delta \\ 0 & \delta \end{pmatrix},$$

and notice that for any nonzero δ, $\beta\delta$ assumes all values in $\mathbb{C}$.

Since N and K are already subgroups of H, the automorphism $\phi : K \to \mathrm{Aut}(N)$ must be given by the conjugation action of K on N, namely

$$\phi(k) = \{n \to knk^{-1}\}.$$

In particular,

$$\phi\left(\begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \right) = \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \delta^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \beta\delta^{-1} \\ 0 & \delta \end{pmatrix} \right\}.$$

(d) Prove that the stabilizer of any nonzero $v \in \mathbb{C}^2$ is isomorphic to this same semidirect product.

Proof. $\mathrm{GL}_2(\mathbb{C})$ acts transitively on $\mathbb{C}^2 - \{(0,0)\}$. Writing $v = (1,0)$, if gv is any other nonzero vector, then $\mathrm{Stab}(gv) = g\mathrm{Stab}(v)g^{-1}$, and conjugate subgroups of the same group are isomorphic.

3. Let G be a non-abelian group with 10 elements.

(a) Prove that G is isomorphic to D_5.

Solution. By Cauchy's theorem, G contains elements r and s of order 5 and 2 respectively, generating subgroups R and S. These intersect trivially, and RS is all of G. (The elements $r^i s^j$ with $0 \le i \le 4$ and $j \in \{0,1\}$ are easily seen to be distinct.)

Moreover, R is normal in G. There are multiple ways to see this. Perhaps the easiest (although not the most direct): the number of 5-Sylow subgroups of G is $\equiv 1 \pmod 5$, and if there were more than 1 then these subgroups would account for more than 10 elements. So the 5-Sylow subgroup is unique, hence normal.

Thus G is isomorphic to a semidirect product $R \rtimes_\phi S$ for a homomorphism $S \to \mathrm{Aut}(R)$ which matches conjugation, i.e., such that $srs^{-1} = \sigma(r)$ where σ is an automorphism of R order 2. The only such automorphisms are the identity and inversion. If it were the identity, we would have $srs^{-1} = r$, so that $sr = rs$, which would imply that r and s commute, and hence that G be abelian. Since this were assumed to not be the case we have $srs^{-1} = r^{-1}$, i.e. $sr = r^{-1}s$.

We therefore have

$$G = \langle r, s : r^5 = s^2 = 1,\ sr = r^{-1}s\rangle,$$

i.e. G is D_5.

(b) G has 5 conjugate subgroups of order 2. (Why?) The action of G by conjugation on these subgroups induces a homomorphism $G \to \mathrm{Sym}(5)$. Describe this homomorphism (including its image and kernel) explicitly.

Solution. The subgroups of order 2 correspond to the five elements of D_5 of order 2. (We know this a priori, but notice that the number of 2-Sylow subgroups is odd, divides 5, and is not 1 since G is abelian).

We represent D_5 as the subgroup of S_5 generated by $r = (1\ 2\ 3\ 4\ 5)$ and $s = (1\ 5)(2\ 4)$, with $rs = sr^{-1}$. Write $s = s_3$, and notice that each conjugate of s fixes exactly one element. These are the five elements of order 2; write s_i for $i \in \{1, 2, 3, 4, 5\}$ for them, where the subscript indicates the fixed element. Specifically, we have

$$s_1 = (2\ 5)(3\ 4),\ s_2 = (1\ 3)(4\ 5),\ s_3 = (1\ 5)(2\ 4),\ s_4 = (1\ 2)(3\ 5),\ s_5 = (1\ 4)(2\ 3).$$

The five 2-Sylow subgroups are of the form $\{1, s_i\}$, so we just need to determine the conjugation action of G on the s_i. We have

$$rs_ir^{-1} = s_{i+1},$$

where $i + 1$ is to be considered modulo 5, because the permutation on the left fixes $i + 1$. So, in our homomorphism $G \to \mathrm{Sym}(5)$, r maps to the permutation $\{s_i \to s_{i+1}\}$.

We can also check that we have

$$s_3s_1s_3^{-1} = s_5,\ s_3s_2s_3^{-1} = s_4,\ s_3s_3s_3^{-1} = s_3,\ s_3s_4s_3^{-1} = s_2,\ s_3s_5s_3^{-1} = s_1.$$

This can be verified directly. The shortest proof observes that, for each i, $s_3s_is_3^{-1}$ has to fix $s_3(x)$, where x is the unique element of $\{1, 2, 3, 4, 5\}$ fixed by s_I.

We therefore observe that the map of ordered sets $\{1, 2, 3, 4, 5\} \to \{s_1, s_2, s_3, s_4, s_5\}$ is such that the action of $D_5 = \langle r, s\rangle$ on $\{1, 2, 3, 4, 5\}$ matches exactly its action by conjugation on $\{s_1, s_2, s_3, s_4, s_5\}$, and on the corresponding 2-Sylow subgroups consisting of 1 and these elements. Therefore, the homomorphism $D_5 \to S_5$ is an isomorphism onto its image.

4. The following question concerns subgroups of $\mathrm{Sym}(p)$, where p is an odd prime.

(a) Let H be a *transitive* subgroup of $\mathrm{Sym}(p)$: that is, for any integers m and n in $\{1, 2, \dots, p\}$ there exists $\sigma \in H$ with $\sigma(m) = n$.

Prove that $p \mid |H|$ and that H contains a p-cycle.

Solution. For each $i \in \{1, 2, \cdots, p\}$ write H_i for the subset of $\sigma \in H$ with $\sigma(1) = i$. Then, choosing any $\sigma_i \in H_i$, we have $H_i = \sigma_iH_1$. So all the H_i are equal in size. Therefore $p \mid |H|$, and H contains an element of order p (necessarily a p-cycle) by Cauchy's Theorem (or Sylow's theorem).

(b) If in addition H contains a transposition, prove that $H = \mathrm{Sym}(p)$.

Solution. This is essentially brute force. Without loss of generality, assume that the p-cycle is $(1\ 2\ 3 \cdots p)$. (This will be true after conjugating H by a suitable element of $\mathrm{Sym}(p)$.) Further, by cyclically permuting the elements if needed, we may assume the transposition is of the form $(1\ 1+k)$ for some $k \geq 1$. (We *cannot* assume $k = 1$.)

If H contains any two-cycle with $b - a \equiv k \pmod p$, then H contains all such two-cycles, because

$$(1\ 2\ 3 \cdots\ p)^i (a\ b)(1\ 2\ 3 \cdots p)^{-i} = (a + i\ b + i),$$

where $a + i$ and $b + i$ are to be interpreted modulo p.

But we have

$$(1\ 1+k)(1+k\ 1+k+j)(1\ 1+k) = (1\ 1+k+j),$$

so if H contains all two-cycles with $b - a \equiv k \pmod p$ and with $b - a \equiv j \pmod p$, then it contains all two-cycles with $b - a \equiv j + k \pmod p$. In other words, the set of k for which H contains all two-cycles of the form $(a\ b)$ for all a, b with $b - a \equiv k \pmod p$ is a nontrivial subgroup of $\mathbb{Z}/p$, hence all of $\mathbb{Z}/p$. So H contains all the transpositions and is hence all of $\mathrm{Sym}(p)$.

(c) Now let K be a *normal* subgroup of $\mathrm{Sym}(p)$, and suppose that K contains a transposition (but don't assume without proof that it contains a p-cycle). Prove that $K = \mathrm{Sym}(p)$.

Solution. All the transpositions are conjugate, hence contained in K (if K is normal). So K contains all the transpositions and is hence all of $\mathrm{Sym}(p)$.