Midterm Exam 1 – Vector Calculus, Frank Thorne (thorne@math.sc.edu)

Due Monday, March 30, 2020 at 9:40 a.m.

Instructions and Advice:

- This is a 2 hour take home exam. Please take it in one continuous 2 hour block of your choosing. (It is recommended that you find somewhere quiet where you can work without interruption.)

- You are allowed to refer to one 4x6 index card during the exam; this may be prepared with anything you like, front and back – as long as it's in your own handwriting and you prepare it yourself.

  Other than that, no books, notes, calculators, cell phones, or assistance from others.

- **Draw pictures, and write complete sentences, where appropriate. Be clear, write neatly, explain what you are doing, and show your work.**

- **Second Chance Opportunity.** After you finish the exam, you are encouraged to use your book, notes, the Internet, etc. to check your work.

  If you find any mistakes, then you may submit a list of corrections and/or new solutions. Please distinguish these from the exam itself (i.e., turn in your exam solutions even if you discover they are wrong). You will get up a bonus of to a quarter of the points you lost.

- The completed exam and Second Chance are due by email (`thorne@math.sc.edu`) to me by 9:40 a.m. on Monday, March 30.

- Once you have taken the exam, please don't talk to anyone (except me) about it before 9:40 a.m. on March 30.

GOOD LUCK!

(1) (15 points) Find an equation for the plane that is perpendicular to the line $x = 3t-5$, $y = 7-2t$, $z = 8-t$ and that passes through the point $(1, -1, 2)$.

(2) (16 points) Explain what it means for a function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ to be differentiable, and give the definition of its derivative. Explain the geometric significance of this definition: what is the derivative describing?

(3) (15 points) Suppose that $\mathbf{f}(x, y) = (x^2, xy, y^2)$.

If $x = r\cos(\theta)$ and $y = r\sin(\theta)$, then define a function $\mathbf{g}(r, \theta) = \mathbf{f}(x(r, \theta), y(r, \theta))$. Compute $D\mathbf{g}$.

(4) (6 points each) A vector field on $\mathbb{R}^3$ is given by

$$\mathbb{F} = 2y\mathbf{i} - 2x\mathbf{j} + z\mathbf{k}.$$

(a) Draw a picture of the $z = 0$ cross section of this vector field (i.e., $2y\mathbf{i} - 2x\mathbf{j}$).

(b) Draw, as best you can, a picture of this vector field in three dimensions. Describe it in words.

(c) Compute its divergence and curl.

(d) Determine at least two different flow lines on this vector field.

(*Hint*: we did not go over any systematic way to do this. This will require a little bit of creativity, as well as an understanding of the geometry of this vector field.)

(5) (6 points each) Suppose that $\mathbf{r} : \mathbb{R} \mapsto \mathbb{R}^3$ is a path describing the motion of a planet around the sun. Let $s(t)$ be the arclength parameter, and let $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ be the unit tangent, normal, and binormal vectors respectively.

(a) Let $\mathbf{v}(t) = \mathbf{r}'(t)$ be the velocity vector. Explain briefly why

$$\mathbf{v}(t) = \frac{ds}{dt}\mathbf{T}.$$

(b) Show that the acceleration vector $\mathbf{a}(t) = \mathbf{r}''(t)$ satisfies

$$\mathbf{a}(t) = \frac{d^2s}{dt}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}.$$

(Recall that $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$.)

(c) By Kepler's first law, the path $\mathbf{r}$ is an ellipse. Draw a typical example (*not* a circle, please), and sketch the vectors $\mathbf{T}$ and $\mathbf{N}$ for several points on the ellipse. What is $\mathbf{B}$ here?

(d) By Kepler's second law, the path $\mathbf{r}$ sweeps out equal areas in equal times. Use this knowledge to indicate on your diagram where $\frac{ds}{dt}$ is positive and where it is negative.

(e) When is $\frac{d^2s}{dt^2}$ positive, and when is it negative? Can you use (b) to give a geometric answer to this question?