Quiz 4 - Math 544, Frank Thorne (thorne@math.sc.edu)

Due Monday, October 26, 2015

(1) A number of two by two matrices are listed. Each of these represents a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. For each linear transformation T:

(i) Compute $T(v)$ for a variety of vectors v, and graph the results.

(ii) Describe the linear transformation in words.

(iii) Draw a cartoon, centered around the origin, in the plane, and draw its image under T.

(iv) Compute the **nullspace** of T: the subspace of vectors $v \in \mathbb{R}^2$ for which $T(v)$ is the zero vector.

(v) Compute the **image** of T: is it the origin only, a line, or all of $\mathbb{R}^2$? (Recall that if the image contains two linearly independent vectors, it is all of $\mathbb{R}^2$.)

(vi) One of the linear transformations represents rotation by 45 degrees. Determine which of them, and prove it.

(a)
$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

(c)
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

(d)
$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

(e)
$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

(f)
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

(g)
$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

(2) Compute the nullspace of the linear transformation represented by the matrix below:

$$\begin{bmatrix} 1 & 1 & -1 \\ 4 & 0 & 4 \\ 2 & 1 & 0 \end{bmatrix}$$

(3) Compute the nullspace of the linear transformation represented by the matrix below:

$$\begin{bmatrix} 1 & 3 & 2 \\ -2 & -6 & -4 \\ 3 & 9 & 6 \end{bmatrix}$$

(4) The image of a linear transformation is equal to the span of the set of columns of the associated matrix. Explain why.

(5) The nullspace of a linear transformation consists only of the zero vector if and only if the columns of the associated matrix are linearly independent. Explain why.

(6) (**Extra Credit 1**) Compute a matrix that represents rotation by a fixed angle θ. Prove that your answer is correct.

(7) (**Extra Credit 2**) Let

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a 2×2 matrix. Prove that all of the following conditions are equivalent to each other:

(a) The two column vectors in the matrix are linearly independent.

(b) The two row vectors in the matrix are linearly independent.

(c) The *determinant* $ad - bc$ does not equal zero.

(d) The nullspace of the associated linear transformation contains only the zero vector.

(e) The range of the associated linear transformation is all of $\mathbb{R}^2$.