Midterm Examination 2 - Math 374, Frank Thorne (thorne@math.sc.edu)

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Please work without books, notes, calculators, or any assistance from others.

(1) (12 points) Give a recursive definition for the set of all strings of well-balanced parentheses.

Example. $(()(()))$ *is such a string, and* $(()))(()$ *is not.*

Solution. Let S be this set. One possible solution is:

- The empty string is in S.
- If a is in S, then so is (a).
- If a and b are in S, then so is ab.

Variants are possible: for example, adding redundant rules is okay. One bonus point for anyone answering in *Backus-Naur form.*

(2) (10 points for equation, 10 points for proof) Find a closed form formula for the recurrence relation given by:

- $T(1) = 1$,
- $T(n) = T(n-1) + n$ for $n \geq 2$.

Use induction to prove that your formula is correct.

Solution. The correct formula is $T(n) = \frac{n(n+1)}{2}$, as one may verify by guess-and-check. The first few values are 1, 3, 6, 10, 15, 21, 28, ... Probably the best way to get started is to compare this to the sequence n^2 or $n^2/2$ and work out how to tweak your guess.

Here is a proof by induction. The formula holds for $n = 1$, because $T(1) = \frac{1(1+1)}{2} = 1$. Assume, for some positive integer k, that

$$T(k) = \frac{k(k+1)}{2}.$$

Then, we have

$$\begin{aligned} T(k+1) &= T(k) + (k+1) \\ &= \frac{k(k+1)}{2} + k + 1 \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+2)(k+1)}{2}, \end{aligned}$$

which verifies the inductive hypothesis for $n = k+1$. The result therefore follows by induction.

(3) (6 points each) What is the cardinality of each of the following sets? (That is, how many elements do they contain?)

(a) $A = \{a, \{b, c\}, \{d\}\}$

(b) $B = \{a, \{a, \{a\}\}\}$

(c) $C = \{\{a\}, \{\{a\}\}\}$

Solution. A contains 3 elements: a, $\{b, c\}$, $\{d\}$. B contains 2 elements: a and $\{a, \{a\}\}$. C contains 2 elements: $\{a\}$ and $\{\{a\}\}$.

(4) (10 points) In a programming language, an identifier must be a single upper-case letter or an upper-case letter followed by a single digit. How many identifiers are possible?

Solution. There are 26 upper case letters and 10 digits, and so a letter may be followed by a digit in $26 \cdot 10 = 260$ ways. The total number of ways is $26 + 260 = 286$.

(5) (8 points each) A survey of 150 college students reveals that 83 own cars, 97 own bicycles, 28 own motorcycles, 53 own a car and a bicycle, 14 own a car and a motorcycle, 7 own a bicycle and a motorcycle, and 2 own all three.

(a) How many students own a bicycle and nothing else?

Solution. Of the 97 students who own bicycles, subtract 53 (who own cars also) and 7 (who own motorcycles also) but then you have to add 2 because you subtracted these students twice.
$97 - 53 - 7 + 2 = 39$.

(b) How many students do not own any of the three?
Solution. By Inclusion-Exclusion, the number of students that own any one of them is

$$83 + 97 + 28 - 53 - 14 - 7 + 2 = 136.$$

So the number of students that own none is $150 - 136 = 14$.

(6) (8 points each) A congressional committee of three is to be chosen from a set of five Democrats and four Republicans.

(a) In how many ways can the committee be chosen?

Solution. An arbitrary choice of 3 out of 9 people: $C(9, 3)$.

(b) In how many ways can the committee be chosen if it must include at least one Democrat?
Solution. There are $C(4, 3)$ committees with only Republicans, so $C(9, 3) - C(4, 3)$.

(c) In how many ways can the committee be chosen if it cannot include both Democrats and Republicans?
Solution. There are $C(4, 3)$ committees with only Republicans, and $C(5, 3)$ with only Democrats, so $C(4, 3) + C(5, 3)$.