3. $$3 + 2 + \frac{4}{3} + \frac{8}{9} + \cdots$$

is a geometric series with

first term $a = 3$

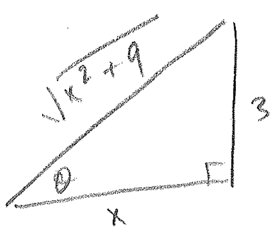
common ratio $r = \frac{2}{3}$.

Because $|r| < 1$ it converges and the value is

$$\frac{a}{1-r} = \frac{3}{1-\frac{2}{3}} = \frac{3}{\frac{1}{3}} = 9.$$

4. $\sum_{n=1}^{\infty} \frac{1}{n^2+9}$ converges if and only if $\int_1^{\infty} \frac{1}{x^2+9}\,dx$ does,

by the integral test.



$$\cot(\theta) = \frac{x}{3}$$

$$\sin(\theta) = \frac{3}{\sqrt{x^2+9}}.$$

So: $x = 3\cot(\theta)$

$\cancel{dx = -3\cot(\theta)\csc(\theta)\,d\theta}$ $dx = -3\csc^2(\theta)\,d\theta.$

This is why it is good to use the quotient rule to check!

$$\sin^2(\theta) = \frac{9}{x^2+9}$$

$$\frac{\sin^2(\theta)}{9} = \frac{1}{x^2+9}$$

and $$\int_1^{\infty} \frac{1}{x^2+9}\,dx = \int_{x=1}^{x=\infty} \frac{\sin^2(\theta)}{9} \cdot \left(-3\,\cancel{\frac{\cos\theta}{\sin\theta}} \cdot \frac{1}{\sin^2\theta}\,d\theta\right)$$

$$= \int_{x=1}^{x=\infty} -\frac{1}{3}\,\cancel{\cos\theta}\,d\theta = -\frac{1}{3}\,\theta\,\Big|_{x=1}^{x=\infty}$$

$$\cancel{-\frac{1}{3\sin\theta}\Big|_{x=1}^{x=\infty}} = -\frac{1}{3}\cot^{-1}\left(\frac{x}{3}\right)\Big|_{x=1}^{x=\infty}.$$

$$\lim_{x\to\infty} \frac{-1}{3}\cot^{-1}\left(\frac{x}{3}\right) + \frac{1}{3}\cot^{-1}\left(\frac{1}{3}\right).$$

What is $\cot^{-1}(\text{BIG})$? cot is $\frac{\text{adj}}{\text{opp}}$

Here $\cot\theta$ is very big.

Drawing a graph or triangle lets us recall that $\lim_{x\to\infty} \frac{-1}{3}\cot^{-1}\left(\frac{x}{3}\right) = 0$.

In particular this integral converges to $\frac{1}{3}\cot^{-1}\left(\frac{1}{3}\right)$ and so does our sum.

Alternate solution:

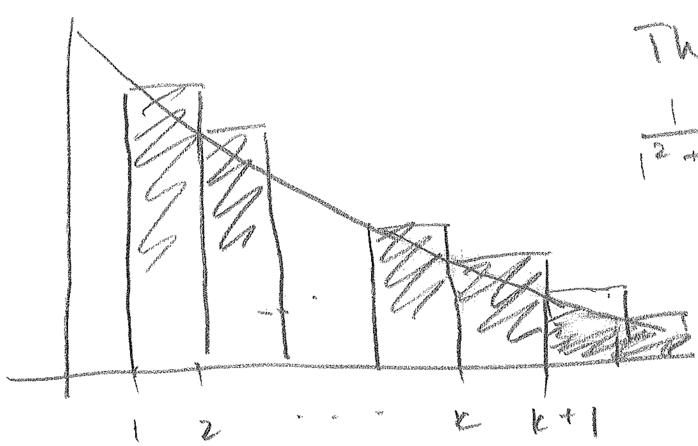
If we drew the triangle differently, would get

$$\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\Bigg|_{x=1}^{x=\infty} = \frac{1}{3}\cdot\frac{\pi}{2} - \frac{1}{3}\cdot\tan^{-1}\left(\frac{1}{3}\right).$$

This is the same thing, which is not obvious.

In either case we know it converges

(b)



The boxes represent

$$\frac{1}{1^2+9} + \cdots + \frac{1}{k^2+9} + \int_{k+1}^{\infty} \frac{1}{x^2+9}\,dx.$$

They are above the shaded area which explains why this (the shaded area) is an ~~over~~ underestimate.

The error is at most $\dfrac{1}{(k+1)^2+9}$.

If $k=1$ then this is $\dfrac{1}{2^2+9} = \dfrac{1}{13} < 1$.

So we can take our <u>lower bound</u>

$$\frac{1}{10} + \int_2^\infty \frac{1}{x^2+9}\,dx = \frac{1}{10} + \frac{1}{3}\cot^{-1}\left(\frac{2}{3}\right)$$

and our <u>upper bound</u>

$$\frac{1}{10} + \frac{1}{23} + \int_2^\infty \frac{1}{x^2+9}\,dx = \frac{1}{10} + \frac{1}{23} + \frac{1}{3}\cot^{-1}\left(\frac{2}{3}\right).$$

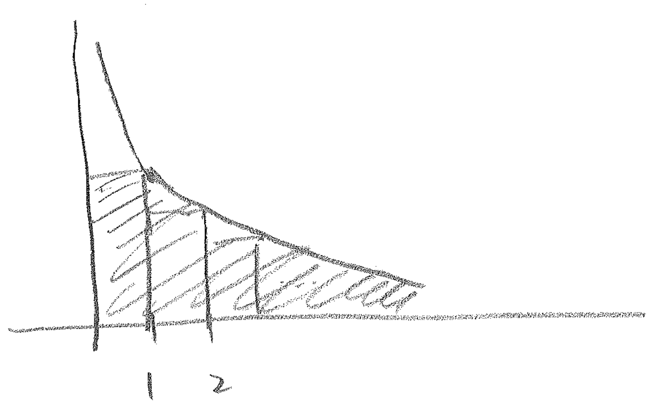
5. Look at $\displaystyle\sum_{n=1}^\infty \frac{n^2-1}{n^4+1}$.

We have $\dfrac{n^2-1}{n^4+1} < \dfrac{n^2}{n^4}$, so $\displaystyle\sum_{n=1}^\infty \frac{n^2-1}{n^4+1} < \sum_{n=1}^\infty \frac{n^2}{n^4}$

$$= \sum_{n=1}^\infty \frac{1}{n^2}.$$

This is a p-series with $p=2$, so since it converges, our original series converges too by the comparison test.

Upper bound for $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2}$:



Our upper bound <u>(with $k=0$)</u> is

$$\frac{1}{1^2} + \int_1^\infty \frac{1}{x^2}\,dx$$

$$= 1 + \left.\frac{-1}{x}\right]_1^\infty = 1 + \left(0-(-1)\right) = \underline{\underline{2}}$$ by the integral test.

7. $\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln(n)}$.

This fails the nth term test, and hence diverges,
(i.e. the nth term test shows it diverges)

because $\lim_{x \to \infty} \frac{x}{\ln(x)} = \lim_{x \to \infty} \frac{\frac{d}{dx}(x)}{\frac{d}{dx}(\ln x)}$ (by L'Hôpital)

$$= \lim_{x \to \infty} \frac{1}{\frac{1}{x}} = \lim_{x \to \infty} x = \infty.$$

Note this does not satisfy condition (2) for the alternating series test.