7.4

18. $\displaystyle\int \frac{x^2+2x-1}{x^3-x}\,dx.$

Factor $x^3 - x = x(x-1)(x+1)$

By partial fractions,

$$\frac{x^2+2x-1}{x^3-x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$x^2+2x-1 = A(x-1)(x+1) + B(x)(x+1) + C(x)(x-1)$$
$$= x^2[A+B+C] + x[B-C] + [-A].$$

So: $A = 1.$

$1 + B + C = 1$, and so $B + C = 0$

$B = -C.$

$B - C = 2$, so $2B = 2.$

$B = 1$ and $C = -1.$

So, $$\frac{x^2+2x-1}{x^3-x} = \frac{1}{x} + \frac{1}{x-1} - \frac{1}{x+1}$$

so $$\int \frac{x^2+2x-1}{x^3-x}\,dx = \int \left(\frac{1}{x} + \frac{1}{x-1} - \frac{1}{x+1}\right)dx$$
$$= \ln|x| + \ln|x-1| - \ln|x+1| + C.$$

6.1, 12.

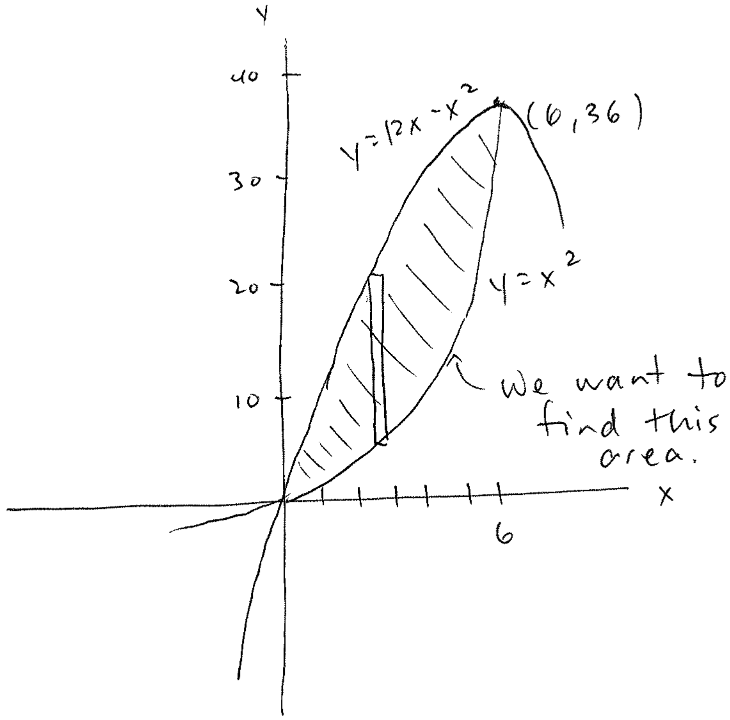
Find the area between $y = x^2$ and $y = 12x - x^2$.

The graph of $y = 12x - x^2$:

If $y' = 0$, $y' = \frac{dy}{dx} = 12 - 2x = 0$

So $x = 6$.

(Top of the parabola)



Intersection points:

$x^2 = 12x - x^2$

$2x^2 - 12x = 0$

$x^2 - 6x = 0$

$x(x-6) = 0$,

so $x = 0$ or $x = 6$.

A typical slice looks like this:

Height is (Top) − (Bottom)

$= (12x - x^2) - x^2$

$= 12x - 2x^2$.

width is dx.

We range from 0 to 6.

So the area is $\int_0^6 (12x - 2x^2)\,dx$

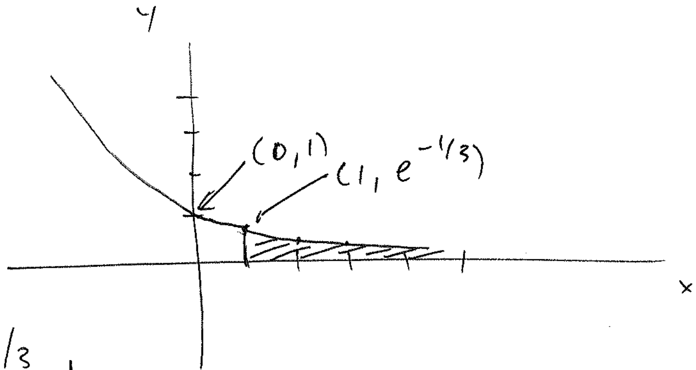
$= 6x^2 - \frac{2}{3}x^3 \Big]_0^6 = 6 \cdot 6^2 - \frac{2}{3} \cdot 6^3 - \left(6 \cdot 0^2 - \frac{2}{3} 0^3\right)$

$= 6^3 - \frac{2}{3} \cdot 6^3$

$= \frac{1}{3} \cdot 6^3 = \frac{1}{3} \cdot 216 = 72$.

This area fits in a box of width 6 and height 36, and $72 = \frac{1}{3} \cdot 216$, so our computation shows that we have filled up one third of the box.

7.8/9. $\int_1^{\infty} e^{-y/3}\, dy.$



This is $\lim\limits_{t\to\infty} \int_1^{t} e^{-y/3}\, dy.$

Set $u = \frac{y}{3}$, $y = 3u$, $dy = 3\,du$

The above is $\lim\limits_{t\to\infty} \int_{y=1}^{y=t} e^{-u} \cdot 3\,du$

$$= \lim_{t\to\infty} \Big[-3e^{-u} \Big]_{y=1}^{y=t}$$

$$= \lim_{t\to\infty} \Big[-3e^{-y/3} \Big]_{y=1}^{y=t}$$

$$= -3\cdot 0 - (-3)\cdot e^{-1/3} = \boxed{3e^{-1/3}}.$$

7.5, 5.

$$\int_1^3 \frac{3t}{(t+1)^2}\,dt.$$

Set $u = t+1$.

Then $du = dt$

$t = u - 1$.

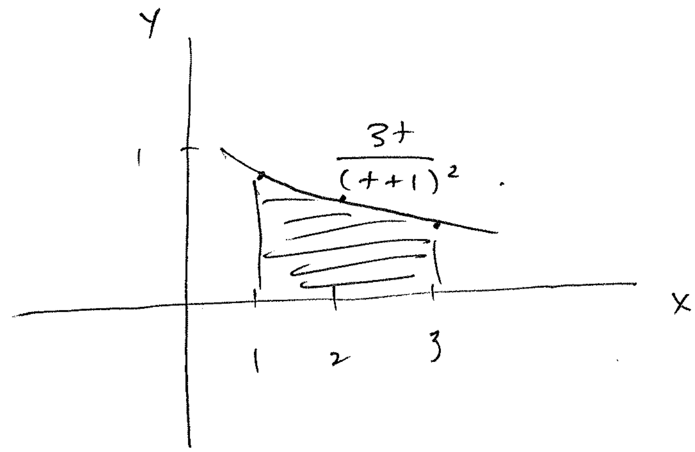
$$\int_{t=1}^{t=3} \frac{3t}{(t+1)^2}\,dt$$

$$= \int_{u=2}^{u=4} \frac{3(u-1)}{u^2}\,du$$

If $t=1$, $\dfrac{3t}{(t+1)^2} = \dfrac{3}{4}$

If $t=2$, $\dfrac{3t}{(t+1)^2} = \dfrac{6}{9} = \dfrac{2}{3}$

If $t=3$, $\dfrac{3t}{(t+1)^2} = \dfrac{9}{16}$.



$$= \int_2^4 \left(\frac{3}{u} - \frac{3}{u^2}\right) du$$

$$= \left[3\ln|u| + \frac{3}{u}\right]_2^4 = \left(3\ln 4 + \frac{3}{4}\right) - \left(3\ln 2 + \frac{3}{2}\right)$$

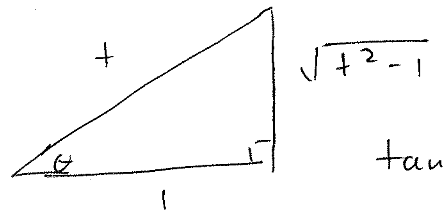
$$= 3\ln 4 - 3\ln 2 - \frac{3}{2} + \frac{3}{4}$$

$$= 3\ln 2 - \frac{3}{4}.$$

This is between 1 and 2.

7.3, 6.

$$\int \frac{\sqrt{t^2-1}}{t}\,dt. \qquad \text{Do a trig substitution.}$$



$$\tan\theta = \sqrt{t^2-1}$$

$$\cos\theta = \frac{1}{t}.$$

So, $t = \sec\theta$

$$dt = \sec\theta \cdot \tan\theta \, d\theta.$$

$$\int \frac{\sqrt{t^2-1}}{t}\,dt = \int \tan\theta \cdot \cos\theta \cdot \sec\theta \cdot \tan\theta \, d\theta$$

$$= \int \frac{\sin\theta}{\cos\theta} \cos\theta \cdot \frac{1}{\cos\theta} \frac{\sin\theta}{\cos\theta}\, d\theta$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta}\, d\theta$$

$$= \int \frac{1-\cos^2\theta}{\cos^2\theta}\, d\theta$$

$$= \int (\sec^2\theta - 1)\, d\theta$$

$$= \tan\theta - \theta + C$$

$$= \sqrt{t^2-1} - \tan^{-1}\left(\sqrt{t^2-1}\right) + C.$$

(7.1, 15) $\int_1^e (\ln x)^2\,dx$.

Set $u = (\ln x)^2$, $dv = dx$

$du = 2 \cdot \ln x \cdot \frac{1}{x}\,dx$, $v = x$.

Then $\int_1^e (\ln x)^2\,dx = (\ln x)^2 \cdot x\Big]_1^e - \int_1^e x \cdot 2\ln x \cdot \frac{1}{x}\,dx$

$= x(\ln x)^2\Big]_1^e - 2\int_1^e \ln x\,dx$.

We evaluate $\int \ln x\,dx$ by parts again.

$u = \ln x$, $dv = dx$

$u = \frac{1}{x}\,dx$ $v = x$.

So $\int_1^e \ln x\,dx = \ln x \cdot x\Big]_1^e - \int_1^e x \cdot \frac{1}{x}\,dx$

$= \ln x \cdot x\Big]_1^e - x\Big]_1^e$.

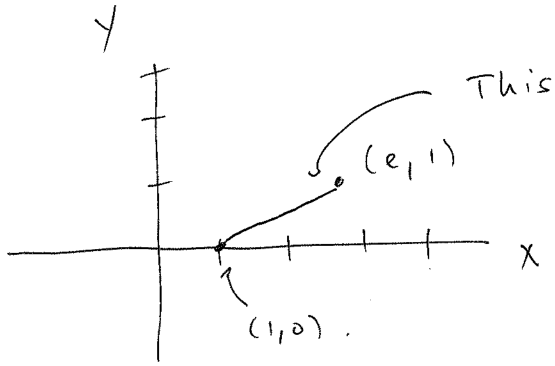
Our original integral is

$$x(\ln x)^2 - 2\ln x \cdot x + 2x\Big]_1^e$$

$$= \left[e(\ln e)^2 - 2\ln(e)\cdot e + 2e\right]$$

$$- \left[x(\ln 1)^2 - 2(\ln 1)\cdot 1 + 2\right]$$

$$= [e - 2e + 2e] - 2 = e - 2.$$



This is concave up but that is not obvious.