Examination 3 - Math 142, Frank Thorne (thorne@math.sc.edu)

Thursday, November 21, 2013

Instructions and Advice:

- You are welcome to as much scratch paper as you need. Turn in everything you want graded, and throw away everything you do not want graded.
- **Draw pictures where appropriate.** If you have any doubt, then a picture is appropriate.
- Be clear, write neatly, explain what you are doing, and show your work. **This is especially important for earning partial credit** in case your work contains one or more mistakes. Be warned that **work I cannot understand will not receive any credit.**
- 75 minutes is a long time. Don't dilly-dally, but don't rush. **You are strongly advised to take the entire 75 minutes to complete the examination.** If you finish early, you have the opportunity to check your work.
- This exam is accompanied by a list of convergence tests which you should freely refer to. Please work without books, notes, calculators, or any assistance from others.
- I will be at the front of the room; if you have any questions, feel free to ask me.

GOOD LUCK!

Convergence Tests for Math 142 — Frank Thorne (thorne@math.sc.edu)

This sheet will be provided to you on the exam.

Advice: I recommend doing the homework while looking at this sheet, but not your notes or book. After you're done, check your notes or book to make sure you didn't make any mistakes.

The convergence tests below concern infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

Sometimes we write $f(n)$ instead of a_n.

Guidelines: 1. We wrote down a series starting at a_1, but in fact the starting value isn't important. If we have a series $\sum_{n=0}^{\infty} a_n$ or $\sum_{n=91887}^{\infty} a_n$ or $\sum_{n=-5189234}^{\infty} a_n$ then the convergence tests work equally well.

2. Only the *eventual behavior* of the series matters. In **all** of the convergence tests below, it is permissible to pick some N and only look at the a_n with $n \geq N$.

3. Some of the convergence tests assume that all of the a_n are nonnegative. They also work if **all** of the a_n are nonpositive, but not necessarily if they have mixed signs.

1. The nth term test. If it is not true that $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

2. The integral test. Suppose that $f(x)$ is a positive, continuous, and decreasing function for $x \geq 1$.

Then, $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_{x=1}^{\infty} f(x)dx$ converges.

Also, if we estimate $\sum_{n=1}^{\infty} f(n)$ by

$$f(1) + f(2) + \cdots + f(k) + \int_{k+1}^{\infty} f(x)dx,$$

then the estimate is too low by somewhere between 0 and $f(k+1)$. (You should know how to draw pictures which explain why this is true.)

3. The alternating series test. Given an alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots,$$

where (1) all the a_i are positive, (2) $a_i > a_{i+1}$ for each i, and $\lim_{n\to\infty} a_n = 0$, then the series converges.

If you approximate the series by the first k terms, the error is between 0 and the $k+1$st term.

4. The comparison test. Suppose you have two series

$$\sum_{n=1}^{\infty} a_n, \quad \sum_{n=1}^{\infty} b_n,$$

where all the a_n and b_n are nonnegative.

- If $a_n \leq b_n$ for each n, and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges and $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.
- If $a_n \geq b_n$ for each n, and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

5. The limit comparison test. Suppose you have two series

$$\sum_{n=1}^{\infty} a_n, \quad \sum_{n=1}^{\infty} b_n,$$

where all the a_n and b_n are nonnegative.

If $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists, and equals some *positive* number other than 0 or ∞, then either both series converge or both series diverge.

6. The absolute convergence test. If a series converges absolutely, then it converges.

7. The ratio test. Suppose that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists and equals some number L.

- If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
- If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(1) (10 points) Does the sequence

$$a_n = \ln(n+2) - \ln(n)$$

converge or diverge? If it converges, find the limit. If it diverges, explain why.

(2) (10 points) Does the sequence

$$a_n = \sqrt{\frac{n+1}{9n+1}}$$

converge or diverge? If it converges, find the limit. If it diverges, explain why.

(3) (10 points) Does the series

$$3 - 4 + \frac{16}{3} - \frac{64}{9} + \cdots$$

converge or diverge? If it converges, find its sum.

(4) (a.) (14 points) By using the integral test, or otherwise, explain why the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 7n + 6}$$

is convergent.

Do only one of (b) and (c). If you turn in both, only (b) will be graded.

(b.) (14 points) Use the integral test to give an upper and a lower bound for the value of this series accurate within 0.1. Draw and explain a graph which represents your lower bound.

(c.) (8 points. Do this if you don't know how to do (b).) Use any method you know to give *any* upper bound for the value of the series.

(5) (14 points) Use the comparison test to determine whether the series converges or diverges. If it converges, determine an explicit upper bound for the value of the series.

$$\sum_{n=1}^{\infty} \frac{9^n}{2 + 10^n}$$

(6) (14 points) Test the alternating series for convergence or divergence. If it converges, determine (*in simplified form*) an estimate for its value which is accurate within 0.25.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{2n^3 - 1}$$

(7) (14 points) Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k$$

$p > 1$ converges

$p < 1$ diverges

$p = 1$ diverges

1. We have $a_n = \ln(n+2) - \ln(n)$

$$= \ln\left(\frac{n+2}{n}\right) = \ln\left(1 + \frac{2}{n}\right).$$

Therefore $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(1+\frac{2}{n}\right) = \lim_{n\to\infty} \ln(1) = 0.$

The sequence converges to 0.

2. We have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{\frac{n+1}{9n+1}} = \lim_{n\to\infty} \sqrt{\frac{\frac{n+1}{n}}{\frac{9n+1}{n}}}$

$$= \lim_{n\to\infty} \sqrt{\frac{1+\frac{1}{n}}{9+\frac{1}{n}}}$$

$$= \sqrt{\frac{1+0}{9+0}} = \sqrt{\frac{1}{9}} = \frac{1}{3}.$$

The sequence converges to $\frac{1}{3}$.

3. This is a geometric series with common ratio $-\frac{4}{3}$.
We have $|r| > 1$. The terms do not approach 0, this fails the nth term test, the series diverges.

4. (a) [Alternate solution: compare to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.]

We have $\frac{1}{n^2+7n+6} = \frac{1}{(n+1)(n+6)}$.

Partial fractions: It equals $\frac{A}{n+1} + \frac{B}{n+6}$,

where $1 = A(n+6) + B(n+1)$,

so, $A = -B$, $6A + B = 1$.

$5A = 1$, so $A = \frac{1}{5}$.

So $\dfrac{1}{(n+1)(n+6)} = \dfrac{1/5}{n+1} - \dfrac{1/5}{n+6}$.

Use the integral test: $\displaystyle\lim_{t\to\infty}\int_1^t \left(\frac{1/5}{x+1} - \frac{1/5}{x+6}\right) dx$

$$= \lim_{t\to\infty} \left(\frac{1}{5}\ln(x+1) - \frac{1}{5}\ln(x+6)\right]_1^t$$

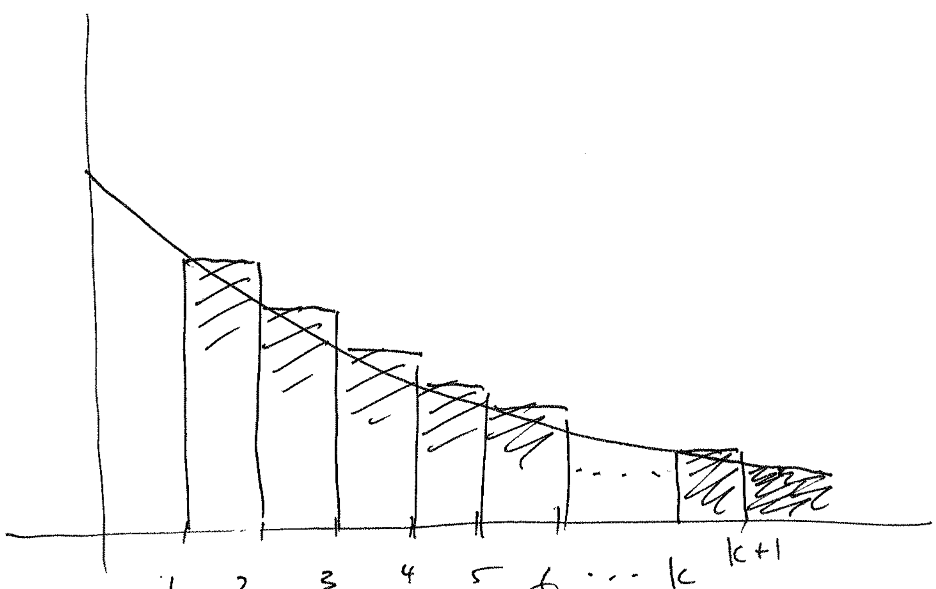
$$= \lim_{t\to\infty} \frac{1}{5}\left[\ln(t+1) - \ln(t+6) - \ln(2) + \ln(7)\right]$$

$$= \lim_{t\to\infty} \frac{1}{5}\left[\ln\left(\frac{t+1}{t+6}\right) + \ln\left(\frac{7}{2}\right)\right]$$

$$= \lim_{t\to\infty} \frac{1}{5}\left[\ln\left(\frac{1+\frac{1}{t}}{1+\frac{6}{t}}\right) + \ln\left(\frac{7}{2}\right)\right]$$

$= \ln(3)/5$. It converges, so the series does also.

(b)



The shaded area is $f(1) + \cdots + f(k) + \displaystyle\int_{k+1}^{\infty} f(x)\,dx$,

where $f(x) = \dfrac{1}{x^2+7x+6}$. (cont.)

We would obtain $f(1) + f(2) + \cdots + f(k) + f(k+1) + \cdots$ if we kept drawing boxes, and the boxes would be above the graph. This is why the graph represents a lower bound.

The upper bound is $f(1) + \cdots + f(k+1) + \int_{k+1}^{\infty} f(x)\,dx$.

The biggest possible error is $f(k+1) = \dfrac{1}{(k+2)(k+7)}$.

If $k=1$ this is already less than $\frac{1}{10}$. In fact we could take <u>any</u> value of k, even $k=0$.

We have $\displaystyle\int_{k+1}^{\infty} f(x)\,dx = \lim_{t \to \infty} \left[\frac{1}{5} \ln\left(\frac{t+1}{t+6}\right) - \frac{1}{5} \ln\left(\frac{k+2}{k+7}\right) \right]$

$$= -\frac{1}{5} \ln\left(\frac{k+2}{k+7}\right) = \frac{1}{5} \ln\left(\frac{k+7}{k+2}\right).$$

If $k=1$ this is $\frac{1}{5} \ln\left(\frac{8}{3}\right)$.

<u>So</u>: lower bound: $f(1) + \frac{1}{5} \ln\left(\frac{8}{3}\right) = \frac{1}{14} + \frac{1}{5} \ln\left(\frac{8}{3}\right)$

upper bound: $f(1) + f(2) + \frac{1}{5} \ln\left(\frac{8}{3}\right)$

$$= \frac{1}{14} + \frac{1}{27} + \frac{1}{5} \ln\left(\frac{8}{3}\right).$$

5. It looks like $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{2+10^n}$.

More specifically, $\displaystyle\frac{9^n}{2+10^n} < \frac{9^n}{10^n}$ for each n.

So, if $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{10^n}$ converges, then so does $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{2+10^n}$,

and $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{2+10^n} < \sum_{n=1}^{\infty} \frac{9^n}{10^n}$.

We have that $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{10^n}$ is a convergent geometric series,

equal to $\displaystyle\frac{9/10}{1-9/10} = \frac{9/10}{1/10} = 9$.

6. We have $\displaystyle\frac{n^2}{2n^3-1} > \frac{(n+1)^2}{2(n+1)^3-1}$ for each n.

(Bonus points if you proved this, either by algebra or by showing $\displaystyle\frac{d}{dx}\left(\frac{x^2}{2x^3-1}\right) < 0$.

So we can use the alternating series test and the series converges.

It is $\displaystyle -\frac{1}{2-1} + \frac{4}{16-1} - \frac{9}{54-1} + \frac{16}{128-1} - \cdots$

$$= -1 + \frac{4}{15} - \frac{9}{53} + \frac{16}{127} - \cdots$$

The third term is less than 0.25,

So the first two terms, $\boxed{\dfrac{-11}{15}}$ are a good estimate.

7. Use the ratio test.

$$\lim_{k \to \infty} \left| \frac{(k+1)\left(\frac{2}{3}\right)^{k+1}}{k \cdot \left(\frac{2}{3}\right)^{k}} \right| = \lim_{k \to \infty} \frac{k+1}{k} \cdot \frac{2}{3}$$

$$= \lim_{k \to \infty} \left(1 + \frac{1}{k}\right) \cdot \frac{2}{3} = 1 \cdot \frac{2}{3} = \frac{2}{3}.$$

This is less then 1, so the series is abcolutely convergent.