Examination 3 - Math 141, Frank Thorne (thornef@mailbox.sc.edu)

Friday, September 20, 2015, 10:50 a.m.

Please work without books, notes, calculators, or any assistance from others. If you have any questions, feel free to ask me. Please do your work on separate paper; you should staple this sheet to your work (put this on top) and turn in everything together.

The first two questions are worth 15 points each. The remaining questions are worth 14 points each.

(1) A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?

Be sure to draw and label a picture.

(2) What is a definite integral? Explain thoroughly and draw a picture.

(3) Solve the initial value problem

$$\frac{dy}{dx} = 9x^2 - 4x + 5, \quad y(-1) = 0.$$

(4) Using rectangles each of whose height is given the value of the function at the midpoint of the rectangle's base (*the midpoint rule*), estimate the area under the graph of the following function, using four rectangles:

$$f(x) = 4 - x^2 \text{ between } x = -2 \text{ and } x = 2.$$

Be sure to draw a picture and simplify your answer.

(5) Graph the function and find its average value over the given interval.

$$f(x) = 3x^2 - 3 \text{ on } [0, 1]$$

(6) Evaluate

$$\int_0^{\pi} (1 + \cos x)\, dx.$$

(7) Evaluate

$$\int 3y\sqrt{7 - 3y^2}\, dy.$$

Examination 3 - Math 141, Frank Thorne (thornef@mailbox.sc.edu)

Friday, September 20, 2015, 12:00 noon

Please work without books, notes, calculators, or any assistance from others. If you have any questions, feel free to ask me. Please do your work on separate paper; you should staple this sheet to your work (put this on top) and turn in everything together.

The first two questions are worth 15 points each. The remaining questions are worth 14 points each.

(1) You are planning to close off a corner of the first quadrant with a line segment 20 units long running from $(a, 0)$ to $(0, b)$. Show that the area of the triangle enclosed by the segment is largest when $a = b$.

Be sure to draw and label a picture.

(2) What is a definite integral? Explain thoroughly and draw a picture.

(3) Solve the initial value problem

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \quad y(4) = 0.$$

(4) Using rectangles each of whose height is given the value of the function at the midpoint of the rectangle's base (*the midpoint rule*), estimate the area under the graph of the following function, using four rectangles:

$$f(x) = x^2 \text{ between } x = 0 \text{ and } x = 1.$$

Be sure to draw a picture and simplify your answer.

(5) Graph the function and find its average value over the given interval.

$$f(t) = t^2 - t \text{ on } [-2, 1]$$

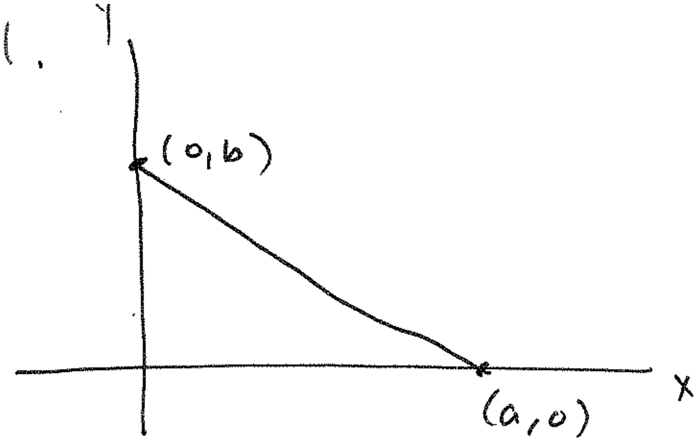
(6) Evaluate

$$\int_0^1 (x^2 + \sqrt{x})\, dx.$$

(7) Evaluate

$$\int \theta \sqrt[4]{1 - \theta^2}\, d\theta.$$

1.



You know $a^2 + b^2 = 20^2 = 400$.

Let $A = \text{area} = \frac{1}{2} ab$.

Want to maximize A,

We have $b^2 = 400 - a^2$, so

$$b = \sqrt{400 - a^2},$$

$$A = \frac{1}{2} a \sqrt{400 - a^2}.$$

Now
$$\frac{dA}{da} = \frac{1}{2} a \cdot \frac{1}{2}(400 - a^2)^{-1/2} \cdot \frac{d}{da}(400 - a^2) + \frac{1}{2}\sqrt{400 - a^2}$$

$$= \frac{1}{4} a (400 - a^2)^{-1/2} \cdot (-2a) + \frac{1}{2}\sqrt{400 - a^2}$$

$$= -\frac{1}{2} a^2 (400 - a^2)^{-1/2} + \frac{1}{2}\sqrt{400 - a^2}$$

$$= -\frac{1}{2} a^2/b + \frac{1}{2} b.$$

This is maximized when $\frac{dA}{da} = 0$, so $\frac{1}{2} b = \frac{1}{2} \frac{a^2}{b}$

$$b^2 = a^2,$$

and hence $\boxed{b = a}$.

(We must have b and a both positive, so $b \neq -a$.)

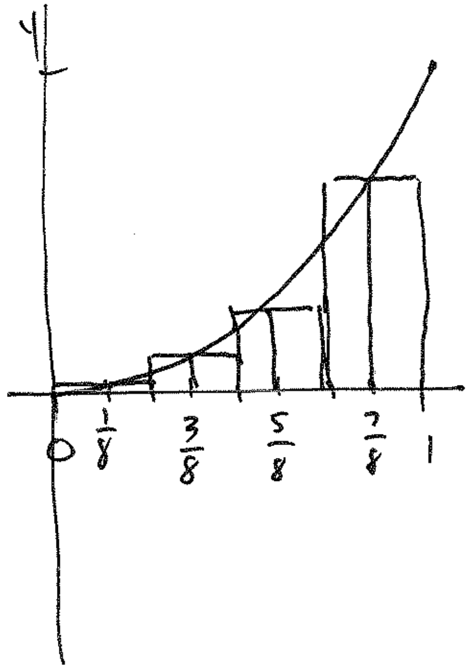
2. See other sheet.

3. $y = \int \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2} \cdot \frac{x^{1/2}}{1/2} + C = x^{1/2} + C$.

Because $y(4) = 0$, $\quad 0 = 4^{1/2} + C = 2 + C$, so $C = -2$,

$$\boxed{y = x^{1/2} - 2}.$$

4.



The area is the area of these four rectangles:

$$\frac{1}{4}\left(\frac{1}{8}\right)^2 + \frac{1}{4}\left(\frac{3}{8}\right)^2 + \frac{1}{4}\left(\frac{5}{8}\right)^2 + \frac{1}{4}\left(\frac{7}{8}\right)^2$$

$$= \frac{1^2 + 3^2 + 5^2 + 7^2}{4 \cdot 8^2}$$

$$= \frac{1 + 9 + 25 + 49}{4 \cdot 8^2}$$

$$= \frac{84}{4 \cdot 8^2} = \frac{21}{8^2} = \frac{21}{64}.$$

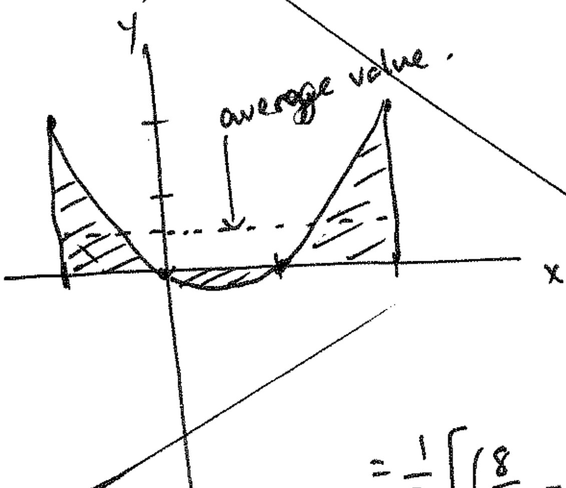
Note that this is very close to $\int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$.

5. $y = t^2 - t = t(t-1)$ has roots at $t = 0, 1$.

$f(2) = 4 - 2 = 2$

$f(-1) = 1 - (-1) = 2$.

Oops! wrong interval



The average value is

$$\frac{1}{2-(-1)} \int_{-1}^{2} (t^2 - t)\,dt$$

$$= \frac{1}{3}\left[\frac{t^3}{3} - \frac{t^2}{2}\right]_{-1}^{2}$$

$$= \frac{1}{3}\left[\left(\frac{2^3}{3} - \frac{2^2}{2}\right) - \left(\frac{(-1)^3}{3} - \frac{(-1)^2}{2}\right)\right]$$

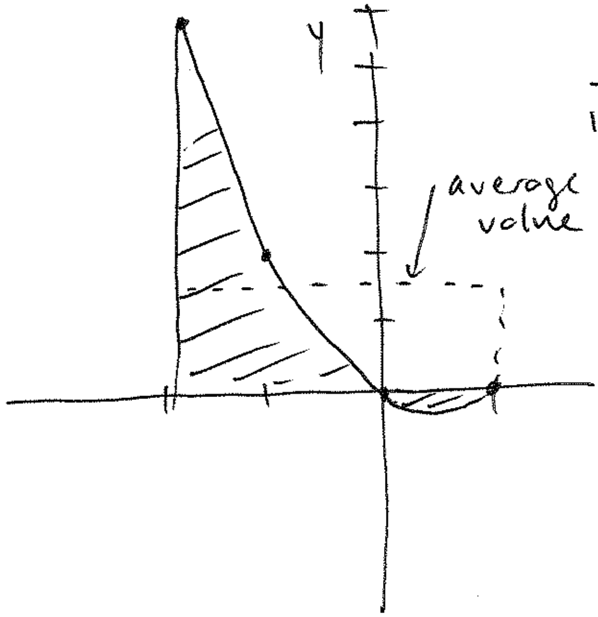
$$= \frac{1}{3}\left[\left(\frac{8}{3} - 2\right) - \left(\frac{-1}{3} - \frac{1}{2}\right)\right] = \frac{1}{3}\left[\frac{8}{3} + \frac{1}{3} - 2 + \frac{1}{2}\right] = \frac{1}{3}\left(3 - \frac{3}{2}\right)$$

$$= \frac{1}{3} \cdot \frac{3}{2} = \boxed{\frac{1}{2}}$$

5. $y = t^2 - t = t(t-1)$ has roots at $t = 0, 1$.

$f(-2) = (-2)^2 - (-2) = 4 + 2 = 6$

$f(1) = 1^2 - 1 = 0$.



The average value is

$$\frac{1}{1-(-2)} \int_{-2}^{1} (t^2 - t)\,dt$$

$$= \frac{1}{3}\left[\frac{t^3}{3} - \frac{t^2}{2}\right]_{-2}^{1}$$

$$= \frac{1}{3}\left[\left(\frac{1}{3} - \frac{1}{2}\right) - \left(\frac{(-2)^3}{3} - \frac{(-2)^2}{2}\right)\right]$$

$$= \frac{1}{3}\left[\left(\frac{1}{3} - \frac{1}{2}\right) - \left(\frac{-8}{3} - \frac{4}{2}\right)\right]$$

$$= \frac{1}{3}\left[\frac{1}{3} - \frac{1}{2} + \frac{8}{3} + \frac{4}{2}\right]$$

$$= \frac{1}{3}\left[\frac{9}{3} + \frac{3}{2}\right] = \frac{1}{3}\left[3 + \frac{3}{2}\right]$$

$$= \frac{1}{3} \cdot \frac{9}{2} = \frac{3}{2}.$$

6. $\int_0^1 (x^2 + \sqrt{x})\,dx = \left[\frac{x^3}{3} + \frac{x^{3/2}}{3/2} \right]_0^1$

$= \left(\frac{1}{3} + \frac{2}{3} \cdot 1^{3/2} \right) - \left(\frac{0^3}{3} + \frac{0^{3/2}}{3/2} \right)$

$= \frac{1}{3} + \frac{2}{3} = 1.$

7. $\int \theta \sqrt[4]{1-\theta^2}\, d\theta$

Set $u = 1 - \theta^2$

$du = -2\theta\, d\theta$

then $\frac{du}{-2} = \theta\, d\theta$

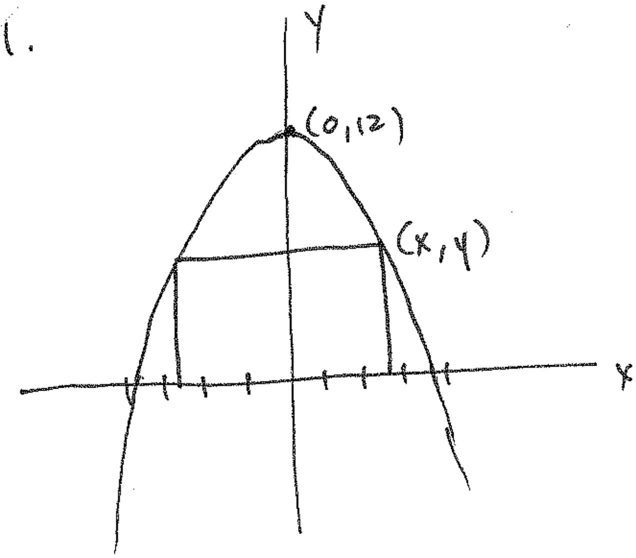
$\int \theta \sqrt[4]{1-\theta^2}\, d\theta = \int \sqrt[4]{u} \cdot \frac{du}{-2} = -\frac{1}{2} \int u^{1/4}\, du$

$= \frac{-1}{2} \cdot \frac{u^{5/4}}{5/4} + C$

$= \frac{-1}{2} \cdot \frac{4}{5}\, u^{5/4} + C$

$= -\frac{2}{5} (1-\theta^2)^{5/4} + C.$

1.



If the top right corner is (x,y) then the rectangle has area $2xy$.

Since $y = 12 - x^2$, this is

$$2xy = 2x(12 - x^2) = 24x - 2x^3.$$

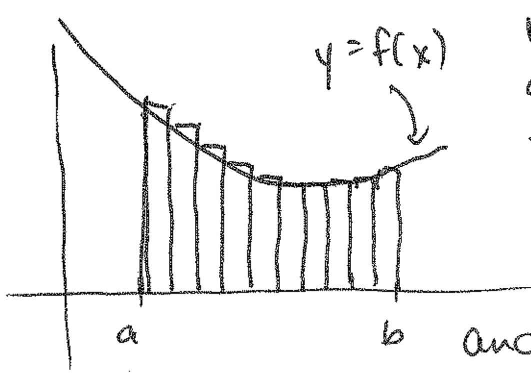
Write area $A(x) = 24x - 2x^3$

~~seck~~ $\frac{dA}{dx} = 24 - 6x^2$

If $\frac{dA}{dx} = 0$ then $\cancel{\boxed{x = 3}}$ $x^2 = 4$, so $\underline{x = 2}$, $y = 12 - 4 = 8$ ~~$12 - 9 = 3$~~

and so the width is 4, the height is 8, and the area is 32.

2.



The definite integral $\int_a^b f(x)\,dx$ represents the area under the graph of $f(x)$ between $x = a$ and $x = b$. It is signed in the sense that when $f(x)$ is below the x-axis we consider the area between $f(x)$ and the x-axis to be negative.

We compute this area by approximating it with boxes of width Δx and taking a limit as $\Delta x \to 0$, so that

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} (\text{Area of boxes})$$

$$= \lim_{\Delta x \to 0} f(a)\Delta x + f(a+\Delta x)\Delta x + f(a+2\Delta x)\Delta x + \dots + f(b-\Delta x)\Delta x.$$

3. If $\frac{dy}{dx} = 9x^2 - 4x + 5$

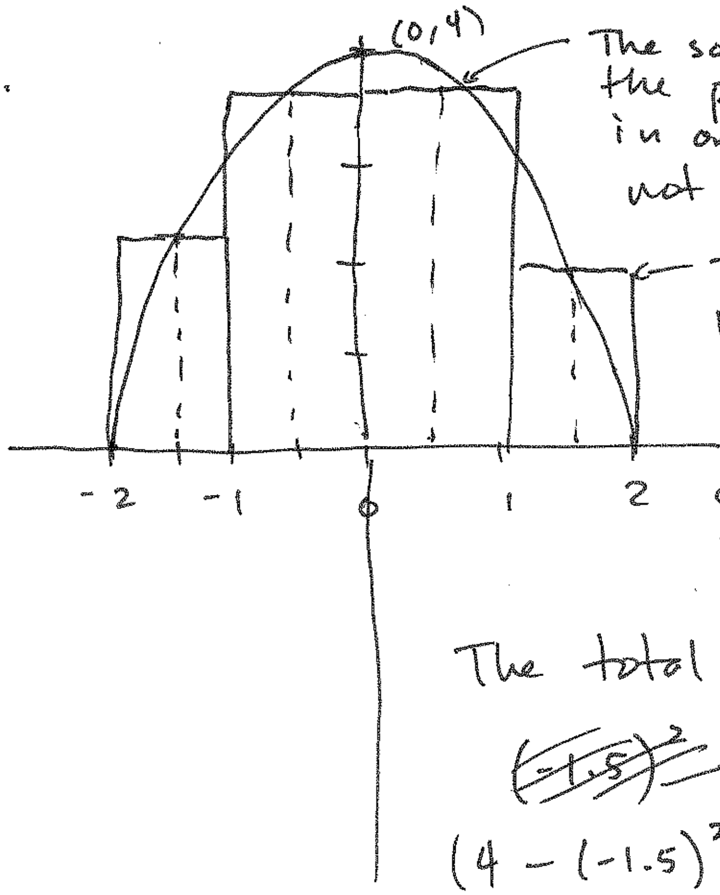
then $y = \int (9x^2 - 4x + 5)\,dx = 3x^3 - 2x^2 + 5x + C$

If $x = -1$, $y = 0$, so $3(-1)^3 - 2(-1)^2 + 5(-1) + C = 0$

$-3 - 2 - 5 + C = 0$, $C = 10$

$$\boxed{y = 3x^3 - 2x^2 + 5x + 10.}$$

4.



The solid line, the dashed line, and the parabola should all intersect in one point but my picture is not totally precise.

This is as tall as the leftmost box.

Each rectangle has base 1, so has area equal to its height.

The total area is

$\cancel{(-1.5)^2 +}$

$(4 - (-1.5)^2) + (4 - (-.5)^2)$

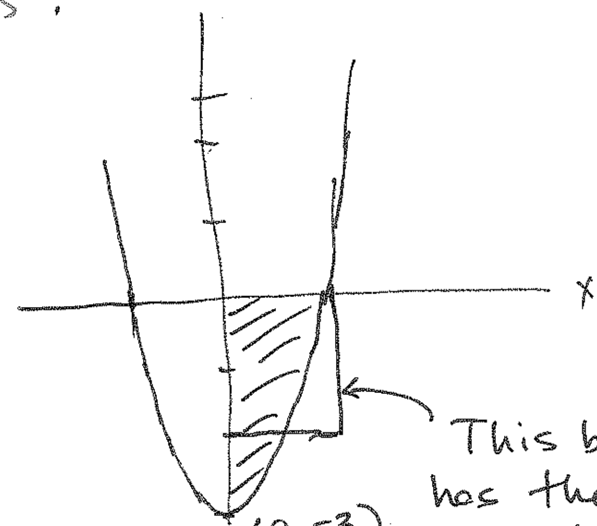
$+ (4 - 0.5^2) + (4 - 1.5^2)$

$= (4 - 2.25) + (4 - 0.25) + (4 - 0.25) + (4 - 2.25)$

$= 1.75 + 3.75 + 3.75 + 1.75 = 11.$

(You can also just compute the area of the first two boxes and double it by symmetry.)

5.



This box has the same signed area as the darkened region.

The average value is

$$\frac{1}{1-0}\cdot\int_0^1 (3x^2-3)\,dx$$

$$= 1\cdot\left[x^3-3x\right]_0^1$$

$$= (1^3-3\cdot 1)-(0^3-3\cdot 0)$$

$$= 1-3-0 = -2.$$

6. $\int_0^\pi (1+\cos x)\,dx = \left[x+\sin x\right]_0^\pi$

$$= (\pi+\sin(\pi)) - (0+\sin(0))$$

$$= \pi+0-(0+0) = \pi.$$

7. Set $u = 7-3y^2$, then $du = -6y\,dy$ so that $3y\,dy = \frac{du}{-2} = -\frac{du}{2}$.

Then

$$\int 3y\sqrt{7-3y^2}\,dy = \int -\frac{du}{2}\cdot u^{1/2} = -\frac{1}{2}\int u^{1/2}\,du$$

$$= -\frac{1}{2}\cdot\frac{2}{3}\cdot u^{3/2}+C$$

$$= -\frac{1}{3}(7-3y^2)^{3/2}+C.$$