Math 575
Problem Set 9

1. Show that if G is a weighted graph and e is an edge whose weight is smaller than that of any other edge, then e must belong to *every* minimum weight spanning tree for G.
   Solution. Suppose that T is a minimum weight spanning tree
   for G that does not contain the edge e.
   Then Consider the graph $T + e$.
   This graph must contain a cycle C that contains the edge e.
   Let f be an edge of C different from e, and set $T^* = T + e - f$.
   Then T^* is also a spanning tree for G, but
   $wt(T^*) = wt(T + e - f) = wt(T) + wt(e) - wt(f) < wt(T)$,
   contrary to T being a minimum weight spanning tree.
   Hence no such tree T (i.e., without e) can exist.

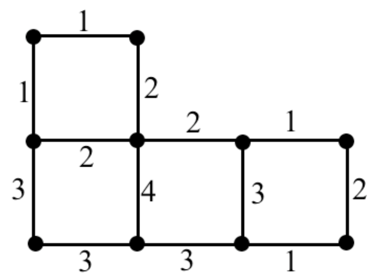
2. Show that if all the weights of the weighted graph G are distinct, then there is a unique minimum weight spanning tree for G.

   Solution. The proof somewhat mimics that of the proof of Kruskal's Algorithm.
   Suppose that T is a tree generated by Kruskal's Algorithm (in fact, a moment's thought shows that with the conditions of the problem, only one such tree could be generated). We claim there are no other minimum weight spanning trees for G. Suppose (and we will show this leads to a contradiction) that there are other minimum weight spanning trees, and choose one, T'.

   Then suppose that e is the *first* edge of T that is not in T'. In other words, suppose that the edges of T, in the order they were added to form T, are $e_1, e_2, \ldots, e_k, \ldots e_{n-1}$ and that $e = e_k$ and for all $i < k$, $e_i \in T'$. Let C be the cycle in $T' + e$ that contains e. let f be an edge of C that is not in T'. We note that by the nature of Kruskal's algorithm, the weight of f must be greater than the weight of e. This is because at the time we placed e into T, f was also available and would not have produced a cycle (since all the edges of T up to that point are in T' as well). So if $wt(f) < wt(e)$, we would have used f at that juncture.

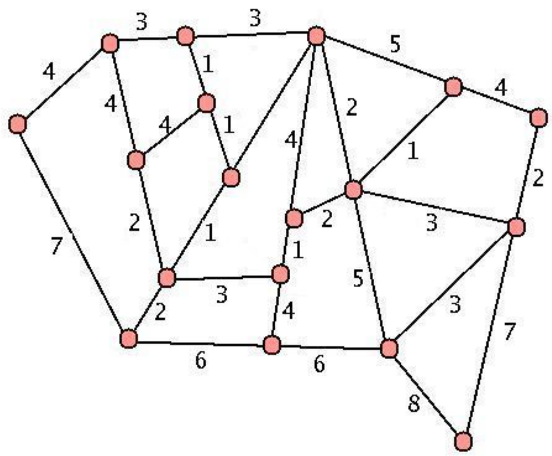
   So now set $T^* = T' + e - f$ is a spanning tree of weight less than T' - a contradiction.

3. Find two distinct minimum weight spanning trees for the graph below.



Solution Any spanning tree of weight 16 is a minimum weight spanning tree.

4. Find a minimum weight spanning tree for the graph below.



5. **Prove**: Every *k*-chromatic graph contains a copy of every tree on *k* vertices.
Proof. Since *G* is a *k*-chromatic graph, *G* contains a subgraph *H* that is *k*-critical (just keep throwing away vertices until you can't do it any longer). Then we know that $\delta(G) \geq k-1$.
Thus *H* (and hence *G* as well) contains every tree on $\delta(G)+1 \geq k$ vertices.

6. **Prove**: If *G* is a connected graph with no induced P_4, then $\chi(G) = \omega(G)$.
Solution. By a previous exercise, we know that $G = A \oplus B$ for some two subgraphs *A* and *B* of *G*. The result now follows by induction and the fact that
$\chi(G) = \chi(A) + \chi(B) = \omega(A) + \omega(B) = \omega(G)$.