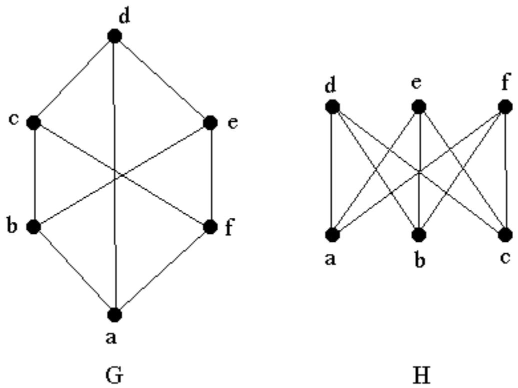
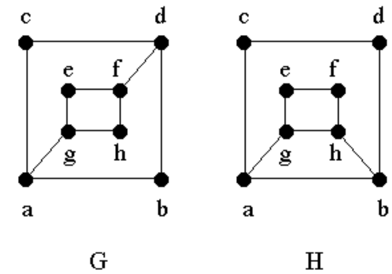
Math 575
Problem Set 4

1. Determine which of the following pairs of graphs are isomorphic. Verify your answers in each case.

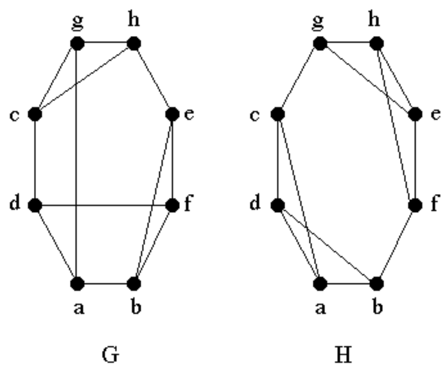
(a).



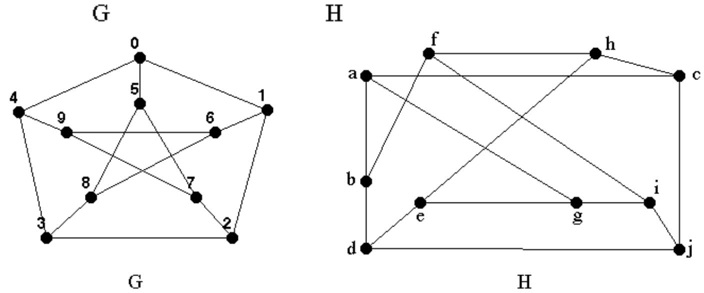
(b).



(c).



(d).



Solution: (a) and (d) are isomorphic pairs. An isomorphism for (d) is:

0	1	2	3	4	5	6	7	8	9
a	c	j	d	b	g	h	i	e	f

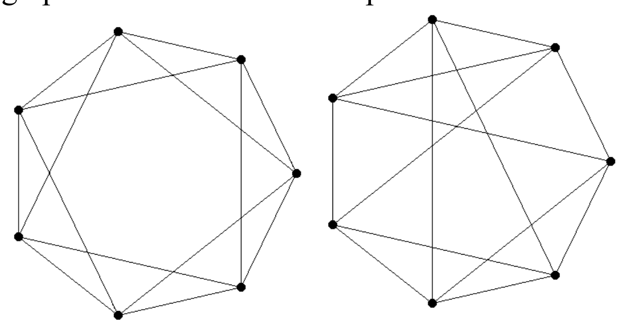
2. **Prove**: If G is isomorphic to H, then $\bar{G}$ is isomorphic to $\bar{H}$.
Solution: If $\phi{:}V(G) \to V(H)$ is an isomorphism from V to G, then it is also an isomorphism from $\bar{G}$ to $\bar{H}$.

3. (a). How many functions from $A = \{1, 2, 3, 4\}$ to the set $B = \{a, b, c, d, e, f\}$ are 1-1?
(b). How many functions from $B = \{a, b, c, d, e, f\}$ to the set $C = \{a, b\}$ are onto?
(b). How many functions from $B = \{a, b, c, d, e, f\}$ to the set $A = \{1, 2, 3, 4\}$ are onto?
[Done in class]

4. Suppose that the number of edges in the vertex-deleted subgraphs of some graph *G* on 10 vertices are 12, 12, 12, 12, 11, 11, 11, 11, 10, 10.
How many edges are there in *G*? What is the degree sequence of *G*?

Answer: 14 edges and the degree sequence is 2, 2, 2, 2, 3, 3, 3, 3, 4, 4

5. Show that the two graphs below are not isomorphic.



Hint: Look at the complements of these graphs and you can't miss.

6. A graph is *self-complementary* if it is isomorphic to its complement. (i.e., $G \cong \overline{G}$)
For example the path P_4 on 4 vertices and the cycle C_5 on five vertices are self-complementary.
Prove: If *G* is self-complementary on *n* vertices, then $n \equiv 1 \bmod 4$ or $n \equiv 0 \bmod 4$.
Hint: Determine an expression for the number of edges that a self-complementary graph must have (in terms of *n*).

Solution: If there are *m* edges in *G*, then since $G \cong \overline{G}$, $\overline{G}$ also has *m* edges. So, $m = \binom{n}{2} - m$, and hence $2m = \binom{n}{2}$ and so $m = \frac{n(n-1)}{4}$. But *m* must be an integer and there you have it.

7. Let *G* be a connected graph on *n* vertices that has no induced path on four vertices.

(a). Show diam(*G*) $\leq$ 2.

Solution: The path between any two vertices a distance 3 or more apart would contain an induced path on four vertices.

(b). Suppose that *v* is a cut-vertex of *G*. Then show that $\deg_G v = n - 1$.

Solution: *Proof*. Suppose that *v* is a cut vertex of *G* and that *G* is connected and has no induced path on four vertices. We must show that *v* is adjacent to every other vertex of *G*. So suppose that $G - v$ has components $A_1, A_2, \ldots, A_k \quad k \geq 2$.
Now assume that *v* is not adjacent to every vertex of $G - v$. We will show that this leads to a contradiction. Then there must be a vertex that is not adjacent to *v*. Without loss of generality, we may assume that such a vertex belongs to the component A_1. Now, since *G* is connected, *v* must be adjacent to some vertex of A_1. So, A_1 contains a vertex adjacent to *v* and it also contains a vertex that is not adjacent to *v*. Thus, by our Differing Neighbors Lemma, there must exist vertices *x* and *y* in A_1 such that $x \perp y$ and $v \perp y$, and $v \not\perp x$ i.e., *x* and *y* are adjacent, *v* and *x* are adjacent, but *v* is not adjacent to *x*.

But now, because *G* is connected, there must be some vertex of A_2 that is adjacent to *v*; let *z* be such a vertex. Then we see that the path *xyvz* is an induced path on four vertices, but this a contradiction and finishes our argument.