

Math 575

Problem Set 14

For any two graphs F and H , the generalized Ramsey number $r(F, H)$ is the smallest integer N such that every coloring of the edges of K_N contains either a red copy of F or a blue copy of H . So for instance, $r(P_n, P_n)$ is the smallest integer N such that every coloring of the edges of K_N contains a monochromatic path on n vertices.

It follows from this definition that N is also the smallest positive integer such that every graph G on N vertices either has F as a subgraph or the complement of G has H as a subgraph. This is because we may view G as the red graph in some red-blue coloring of the edges of K_N and the complement of G as the blue graph in the same coloring.

Example: $r(P_4, P_4) \leq 6$.

Proof. Consider any red - blue coloring of edges of K_6 . Since $r(3, 3) = 6$, we know that there is a red or blue copy of K_3 in this coloring. Let a, b, c denote the vertices of this monochromatic triangle and WLOG assume that its edges are red. Let x, y, z denote the remaining three vertices. Then if any edge from $\{x, y, z\}$ to $\{a, b, c\}$ is red, we have a red P_4 . On the other hand, if none of the edges between $\{x, y, z\}$ and $\{a, b, c\}$ are red, then we have a blue $K_{3,3}$, which contains a blue P_4 within it.

In fact, $r(P_4, P_4) \leq 5$ and we can see this via the following argument.

Consider any red - blue coloring of edges of K_5 . Then for any vertex v in K_5 , some two of its edges, say va and vb have the same color and hence avb is a monochromatic P_3 . WLOG, assume that avb is a red P_3 . Let x and y be the other two vertices of this K_5 . Then if any one of the edges xa, xb, ya, yb are red then together with avb it would form a red P_4 . But if all of these edge are blue then $axby$ is a blue P_4 .

To see that $r(P_4, P_4) > 4$ (and hence that $r(P_4, P_4) = 5$), just note that if you color any triangle of K_4 red and the other three edges blue, then there is no monochromatic P_4 in this coloring.

1. Given that $r(P_4, P_4) = 5$, show that $r(P_5, P_5) = 6$.

2. Show that $r(P_4, C_4) = 5$.

3. Recall the theorem of Erdős that states that for $2 \leq n \leq m$, $r(n, m) \leq r(n-1, m) + r(n, m-1)$.

(a). Use this to show that for any integer $n \geq 2$, $r(n, n) \leq 2r(n-1, n)$.

(b). Show that for $n \geq 2$, $r(3, n) \leq r(3, n-1) + n$.

(c). Show that for $n \geq 2$, $r(3, n) \leq \binom{n+1}{2}$.

4. If F is a graph and k is a positive integer, then by kF we mean k disjoint copies of F .

So, $2K_2$ is the graph that consists of two disjoint edges.

(a). Prove: $r(2K_2, 2K_2) = 5$.

(b). Prove: $r(K_{1,3}, K_{1,3}) = 6$.

(c). Prove: $r(K_{1,4}, K_{1,4}) = 7$.