Hamiltonian and Eulerian Graphs

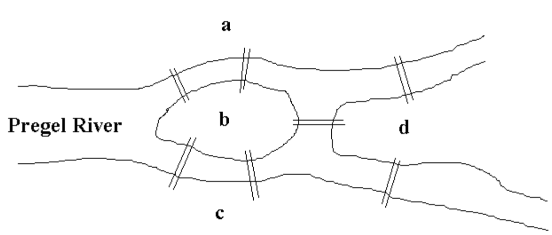
Eulerian Graphs

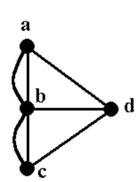
If G has a trail $v_1, v_2, \ldots v_k$ so that each edge of G is represented exactly once in the trail, then we call the resulting trail an *Eulerian Trail.* If the trail is really a circuit, then we say it is an *Eulerian Circuit*. A graph is said to be *Eulerian* if it contains an Eulerian circuit.

The following theorem characterizes Eulerian graphs. It is true for multigraphs as well as graphs.

Theorem (Euler). A connected graph is Eulerian if and only of each vertex has even degree.

This comes from the famous *Seven Bridges of Königsberg* problem where it was asked if it was possible to cross each bridge exactly once and end up where you started. Thus the question is whether the corresponding multigraph is Eulerian.





Exercise: Show half of the proof of Euler's Theorem by showing that if G is Eulerian, then all of its vertices have even degree.

Assuming the truth of Euler's Theorem, verify the corollary below.

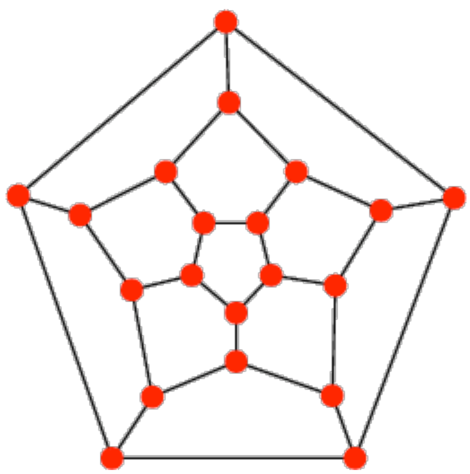
Corollary. A non-trivial connected graph *G* has an Eulerian trail iff it contains exactly two odd vertices. Moreover, in this case, the trail's terminal vertices are the two odd vertices.

These results make it easy to determine that certain puzzles of the variety, "Can you trace the figure below without lifting your pencil from the paper?" are impossible to do.

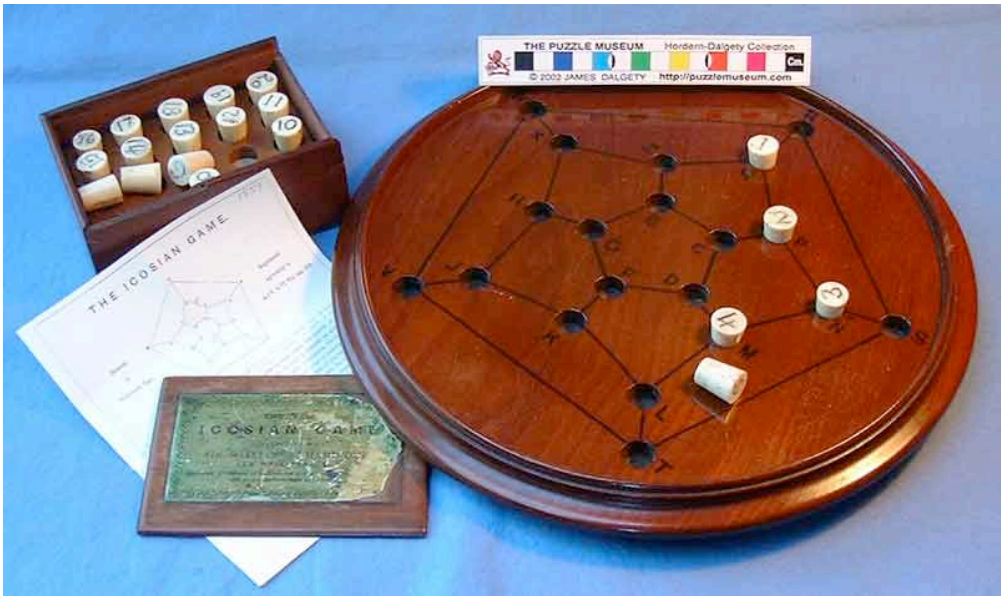
We will concentrate our attention on a more studied notion of Hamiltonian Graphs.

Hamiltonian Graphs

A spanning cycle in a graph is called a *Hamiltonian cycle*, and a spanning path is called a *Hamiltonian path*. A graph is said to be *Hamiltonian* if it has a spanning cycle and it is said to be *traceable* if it has a Hamiltonian path.



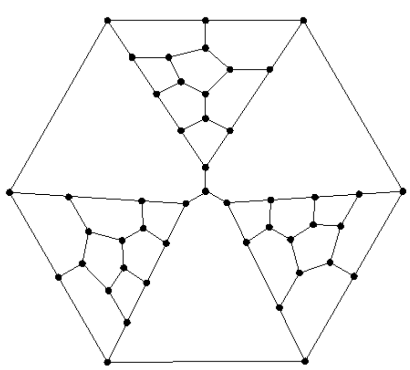
The graph above, known as the *dodecahedron*, was the basis for a game concocted by Hamilton which he dubbed the Around the World Game. Each vertex was given the name of a world city and the object was to visit each city and return to your starting point without ever visiting the same city twice. A picture of the actual puzzle is shown below.

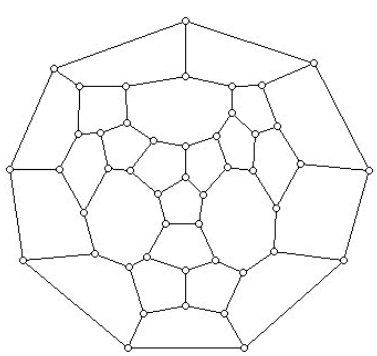


Thus the problem is to find a Hamiltonian cycle in the dodecahedron graph. This is not a particularly challenging thing to do, and the puzzle was not a financial success.

Determining whether a graph has a Hamiltonian cycle can be a very difficult problem and there is no good characterization for Hamiltonian graphs.

To appreciate the problem, the Petersen graph, and the two graphs below are not Hamiltonian and we will soon see a proof of this. But at this point, can you concoct an argument to show why any of these three graphs are not Hamiltonian?





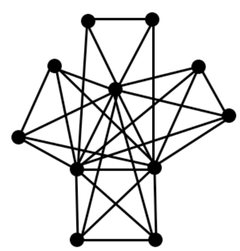
Note that a Hamiltonian graph is clearly 2-connected. Is the converse true? No, and in fact there are many simple examples of 2-connected graphs that are not Hamiltonian.

A 2-connected bipartite graph of odd order would be such an example. More generally, any complete bipartite graph $K_{r,s}$ $2 \le r < s$ would be an example.

These examples suggest a somewhat more general idea which we state next as a theorem. We let $k(H)$ denote the number of components in the graph *H*.

Theorem. If a connected graph *G* contains a set *S* of vertices such that $k(G - S) > |S|$, then *G* is not Hamiltonian.

Example: The graph below is not Hamiltonian since we could choose the set *S* to be the three vertices in the center, and then $G - S$ would have four components.



One immediate and useful consequence of the previous theorem is the result below.

Theorem. If *G* is a connected bipartite graph having bipartition $\{A, B\}$ and $|A| \ne |B|$, then *G* is not Hamiltonian.

[As an exercise, prove this.]

Thus if we can properly color the vertices of a connected graph red and blue and the number of red vertices is different from the number of blue vertices, then the graph is not Hamiltonian.

We can also see that this is true without using the previous theorem, since if a bipartite graph is Hamiltonian and is properly colored red and blue, then its Hamiltonian cycle must be of even order and every consecutive pair of vertices will be red and blue.

The Petersen graph shows that it is also the case that a 3-connected graph need not be Hamiltonian. In fact, no 'amount of connectivity' will suffice to make a graph Hamiltonian.

Exercise. Show that for any positive integer k, there is a k-connected graph that is not Hamiltonian.

Since there is no good characterization for Hamiltonian graphs, we must content ourselves with criteria for a graph to be Hamiltonian and criteria for a graph not to be Hamiltonian.

Among the most fundamental criteria that guarantee a graph to be Hamiltonian are degree conditions. We discuss a few of these next.

To make the explanations a bit smother, we will adopt the following conventions. Let P be a v-u path in a graph G. For any vertex x of P, with $x \neq v$, let x^- denote the vertex of P that precedes x, and similarly if $x \neq u$, let x^+ denote the vertex of P that follows x in P. We will adopt the same conventions for a cycle that is traversed in a prescribed direction (either clockwise or counter-clockwise).

Theorem (Bondy and Chvátal). Suppose that G is a graph and v and u are non-adjacent vertices of G such that $\deg(v) + \deg(u) \geq n$. Then G is Hamiltonian iff $G + vu$ is Hamiltonian.

Proof. Let $e = vu$. Certainly if G is Hamiltonian, then $G + e$ is Hamiltonian. So now suppose that $G + e$ is Hamiltonian but G is not Hamiltonian. In this case, any Hamiltonian cycle C in $G + e$ must contain the edge e, and hence $C - e$ is a v-u path in G. So let P be such a spanning v-u path in G. Suppose that $\deg(v) = k$.

Now we claim that for each $x \in N(v)$, $x^- \notin N(u)$.

For suppose that there is some such vertex x. Then we may follow P from v to x^-, then take the edge x^-u, then follow P in reverse from u to x, then take the edge xv and we have a Hamiltonian cycle in G contrary to our assumption that G is not Hamiltonian. Thus we have established our claim.

So now it must be that $\deg(u) \leq (n - 1) - k = n - k - 1$, and so $\deg(v) + \deg(u) \leq k + n - k - 1 = n - 1$, contrary to our initial assumptions and so the result follows.

For any graph G on n vertices, we define the *Hamiltonian closure* of G (or in this context, just the *closure* of G) to be the graph obtained by recursively joining by an edge any two non-adjacent vertices v and u that satisfy the condition that $\deg(v) + \deg(u) \geq n$. It is not difficult to show that the closure of a graph is unique.

The next result is an immediate consequence of our previous Theorem.

Corollary. If the Hamiltonian closure of a graph is Hamiltonian, then G is Hamiltonian.

Corollary (Ore**)**. If G is a graph on $n \geq 3$ vertices and for every two non-adjacent vertices v and u, $\deg(v) + \deg(u) \geq n$, then G is Hamiltonian.

Corollary. If G is a graph on $n \geq 3$ vertices and for every vertex v $\deg(v) \geq \frac{n}{2}$, then G is Hamiltonian.

Here is a much different kind of condition due to Erdös and Chvátal.

Theorem. Let G be a graph on $n \geq 3$ vertices such that $2 \leq \beta(G) \leq \kappa(G) = k$. Then G is Hamiltonian.

Proof. Suppose that G is not Hamiltonian (and we hope to arrive at a contradiction). Choose a longest cycle C in G (how do you know G has a cycle as all?). Then since C cannot contain be a Hamiltonian cycle for G, there is some vertex v not on C. Since G is k-connected, then by the Fan Lemma there exist k internally disjoint paths $P_1, P_2, \ldots, P_k$ from v terminating in C and we may assume that no interior vertex of any P_i belongs to C.

For each $1 \leq i \leq k$, let x_i be the terminal vertex of P_i. We first note that for each $1 \leq\ i < k$, x_i is not adjacent to x_{i+1} as otherwise we would easily get a longer cycle than C by appending the paths P_i, P_{i+1} and removing the edge $x_i x_{i+1}$.

Let $S = \{x_1^-, x_2^-, x_3^-, \ldots, x_k^-, v\}$. Then since S has cardinality greater than $\beta(G)$, it follows that S cannot be an independent set in G.

But if v is adjacent to x_i^-, then follow v along P_i to x_i then follow C from x_i

to x_i^- clockwise along *C*, and then take the edge $x_i^- v$ and we have produced a Hamiltonian cycle in *G*.

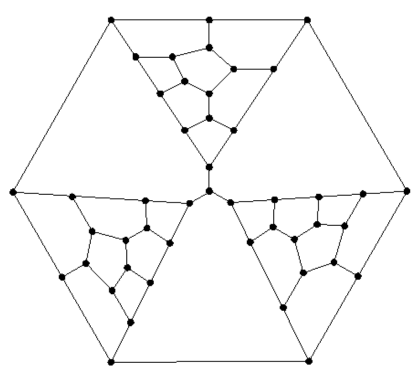
So it must be that for some $i \neq j$, x_i^- is adajcent to x_j^-. But then we get a Hamiltonian cycle by starting at v, taking the path P_i to x_i, then traversing *C* in a clockwise fashion to x_j^-, then taking the edge $x_j^- x_i^-$, then following *C* in a counter-clockwise direction from x_i^- to x_j, and finally following P_j in reverse to *v*. Thus the theorem follows.

Next we turn our attention to Hamiltonian cycles in planar graphs.

There is a very famous conjecture due to the mathematician Tait that asserts that every cubic, 3-connected, planar graph must be Hamiltonian. This result, if true, would be remarkable because – as Tait demonstrated – the Four Color Theorem would follow from it.

Conjecture (Tait). Every 3-connected, planar graph is Hamiltonian.

It turns out, unfortunately, that this conjecture is false. But it is quite difficult to find a counter-example. The first counter-example was provided by Tutte (from the University of Waterloo) and his proof that it was a counter-example was complex. His counterexample appears below – it is called the *Tutte Graph*.



We will soon see why this graph is not Hamiltonian via a simpler (but still non-trivial) argument than that used by Tutte. For now, you should convince yourself that the graph is 3-connected and planar.

A much simpler example along with a very powerful non-Hamiltonian condition was later supplied by the Russian mathematician Grinberg. We will state his result after first defining a few terms.

Suppose G is a plane graph that has a Hamiltonian cycle C. We refer to a region K as being *outside* C if it is possible to join a vertex inside K to a point in the exterior region by an unbroken curve that does not meet C.

A *diagonal* (aka *chord*) of a cycle is an edge that joins two non-consecutive vertices of the cycle.

Theorem (Grinberg). Let G be a plane graph on n vertices that has a Hamiltonian cycle C. Let f_i denote the number of i-regions inside C and f_i' the number of i-regions outside C, and let $\Delta f_i = f_i - f_i'$. Then $\sum_{i=3}^{n}(i-2)\Delta f_i = 0$.

Proof. Let d denote the number of diagonals inside C. Then there are $d + 1$ regions inside C. So, $\sum_{i=3}^{n} f_i = d+1$.

Also, letting I denote the set of inside regions of G, $\sum_{i=3}^{n} if_i = \sum_{R\in I}^{n}|\partial(R)| = n+2d$.

Hence, $\sum_{i=3}^{n}(i-2)f_i = n-2$. Similarly, $\sum_{i=3}^{n}(i-2)f'_i = n-2$.

And so, $\sum_{i=3}^{n}(i-2)(f_i - f_i') = 0$.

Grinberg's Theorem gives a necessary condition for a graph to be planar. You will typically use it to show that a graph is *not* planar or to infer some properties of a known Hamiltonian cycle in the graph.