An Introduction to Bipartite Graphs

If $P$ is a path from the vertex $v$ to the vertex $u$, we refer to $P$ as a $v$-$u$ path (or often just a $vu$-path). If $P$ is a $v$-$u$ path, say $v = v_0v_1v_2\ldots v_k \ldots v_m = u$, then we refer to $v_i v_{i+1} \ldots v_j$ (for any $0 \leq i < j \leq m$) as the $v_i$-$v_j$ subpath of $P$. A shortest $v$-$u$ path is called a $v$-$u$ geodesic.

Note that if the path $P: v = v_0v_1v_2\ldots v_k \ldots v_m = u$ is a $v$-$u$ geodesic, then for every $0 \leq i \leq m$, $d(v, v_i) = i$, and in particular the length of a $v$-$u$ geodesic is $d(v, u)$, the distance from $v$ to $u$. Also, for any such $v$-$u$ geodesic, $d(v_i, u) = m - i$. Thus if $x$ is any vertex on $P$, the $v$-$x$ subpath of $P$ is a shortest $v$-$x$ path, and the $x$-$u$ subpath of $P$ is a shortest $x$-$u$ path. Thus $x = v_j$ where $d(v, x) = j$.

A set $S$ of vertices of a graph $G$ is said to be independent if no two vertices of $S$ are adjacent. Also, we refer to the subgraph induced by $S$ as an independent subgraph. Similarly, $S$ is said to be complete if every two vertices of $S$ are adjacent, and we refer to the subgraph induced by $S$ as a complete subgraph.

**Definition.** A graph $G$ is bipartite if it is the trivial graph or if its vertex set can be partitioned into two independent, non-empty sets $A$ and $B$. We refer to $\{A, B\}$ as a bipartition of $V(G)$.

**Note:** Some people require a bipartite graph to be non-trivial.

Examples include any even cycle, any tree, and the graph below.

![Bipartite Graph Example](image)

**A Few Observations**

(i). No odd cycle is bipartite.
(ii). Trees are bipartite.
(iii). If $G$ is bipartite, then so is every subgraph of $G$.
(iv). If $G$ is bipartite, then it is possible to assign colors red and blue to the vertices of $G$ in such a way, that no two vertices of the same color are adjacent.
(v). $G$ is bipartite if and only if each of its components is bipartite.
Theorem. A graph $G$ is bipartite if and only if it has no odd cycles.

Proof. First, suppose that $G$ is bipartite. Then since every subgraph of $G$ is also bipartite, and since odd cycles are not bipartite, $G$ cannot contain an odd cycle. That’s the easy direction.

Now suppose that $G$ is a non-trivial graph that has no odd cycles. We must show that $G$ is bipartite. So we must determine a partition of the vertices of $G$ into independent sets.

It is enough to prove our result for connected graphs since if $G$ is bipartite, so is every component of $G$ (and vice versa).

So, now consider any vertex $a$ of $G$. Let $A = \{v : d(v, a) \text{ is even}\}$. Similarly, define $B = \{v : d(v, a) \text{ is odd}\}$. Clearly then $V(G) = A \cup B$. We will be finished if we can show that $A$ and $B$ are independent sets.

So we assume that $A$ is not independent and show that this leads to a contradiction. Suppose that $x$ and $y$ are adjacent vertices of $A$. We may assume that for some integers $k, m$ that $d(a, x) = 2k$, and $d(a, y) = 2m$.

Now let $P$ be a shortest $a$-$x$ path, and $Q$ a shortest $a$-$y$ path. Say $P$ is $a = v_0 v_1 v_2 \ldots v_{2k} = x$ and $Q$ is $a = u_0 u_1 u_2 \ldots u_{2m} = y$.

We might notice here that $y$ cannot be on $P$ and $x$ cannot be on $Q$. (Be sure that you can explain why this is true.)

Let $w$ be the vertex in $V(P) \cap V(Q)$ that is closest to $x$.

So, $w = v_j = u_j$ where $d(a, w) = j$. So now consider $P'$, the $w$-$x$ subpath of $P$, and $Q'$, the $w$-$y$ subpath of $Q$. Then $V(P') \cap V(Q') = \{w\}$.

But then the cycle formed by following $P'$ from $w$ to $x$, then the edge $xy$, and then following $Q'$ in reverse from $y$ to $w$ is an odd cycle; more precisely, the cycle $w = v_j v_{j+1} v_{j+2} \ldots v_{2k-1} xy u_{2m-1} u_{2m-2} \ldots w$ has length $(2k - j + (2m - j) + 1 = 2(k + m - j) + 1$, which is odd.

But this contradicts the assumption that $G$ has no odd cycles. Thus it must be that $A$ is independent. A similar argument shows that $B$ is independent.

So our result is proven.