An Introduction to Bipartite Graphs

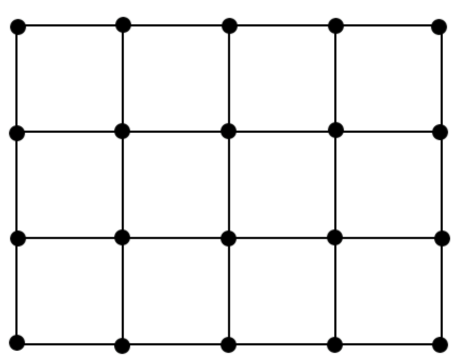
If P is a path from the vertex v to the vertex u, we refer to P as a v-u path (or often just a vu-path). If P is a v-u path, say $v = v_0v_1v_2\ldots v_k\ldots v_m = u$, then we refer to $v_iv_{i+1}\ldots v_j$ (for any $0 \le i < j \le m$) as the v_i-v_j subpath of P. A shortest v-u path is called a v-u *geodesic*.

Note that if the path $P: v = v_0v_1v_2\ldots v_k\ldots v_m = u$ is a v-u geodesic, then for every $0 \le i \le m$, $d(v, v_i) = i$, and in particular the length of a v-u geodesic is $d(v,u)$, the distance from v to u. Also, for any such v-u geodesic, $d(v_i, u) = m - i$. Thus if x is any vertex on P, the v-x subpath of P is a shortest v-x path, and the x-u subpath of P is a shortest x-u path. Thus $x = v_j$ where $d(v, x) = j$.

A set S of vertices of a graph G is said to be *independent* if no two vertices of S are adjacent. Also, we refer to the subgraph induced by S as an independent subgraph. Similarly, S is said to be *complete* if every two vertices of S are adjacent, and we refer to the subgraph induced by S as a *complete subgraph*.

Definition. A graph G is *bipartite* if it is the trivial graph or if its vertex set can be partitioned into two independent, non-empty sets A and B.
We refer to $\{A, B\}$ as a *bipartiton* of $V(G)$.
Note: Some people require a bipartite graph to be non-trivial.

Examples include any even cycle, any tree, and the graph below.



A Few Observations
(i). No odd cycle is bipartite.
(ii). Trees are bipartite.
(iii). If G is bipartite, then so is every subgraph of G.
(iv). If G is bipartite, then it is possible to assign colors red and blue to the vertices of G in such a way, that no two vertices of the same color are adjacent.
(v). G is bipartite if and only if each of its components is bipartite.

Theorem. A graph G is bipartite if and only if it has no odd cycles.
Proof. First, suppose that G is bipartite. Then since every subgraph of G is also bipartite, and since odd cycles are not bipartite, G cannot contain an odd cycle. That's the easy direction.

Now suppose that G is a non-trivial graph that has no odd cycles. We must show that G is bipartite. So we must determine a partition of the vertices of G into independent sets.
It is enough to prove our result for connected graphs since if G is bipartite, so is every component of G (and vice versa).

So, now consider any vertex a of G. Let $A = \{v : d(v,a) \text{ is even}\}$. Similarly, define, $B = \{v : d(v,a) \text{ is odd}\}$. Clearly then $V(G) = A \dot{\cup} B$. We will be finished if we can show that A and B are independent sets.

So we assume that A is not independent and show that this leads to a contradiction.
Suppose that x and y are adjacent vertices of A. We may assume that for some integers k, m that $d(a,x) = 2k$, and $d(a,y) = 2m$.

Now let P be a shortest a-x path, and Q a shortest a-y path.
Say P is $a = v_0 v_1 v_2 \ldots v_{2k} = x$ and Q is $a = u_0 u_1 u_2 \ldots u_{2m} = y$.

We might notice here that y cannot be on P and x cannot be on Q . (Be sure that you can explain why this is true.)

Let w be the vertex in $V(P) \cap V(Q\}$ that is closest to x.

So, $w = v_j = u_j$ where $d(a,w) = j$. So now consider P', the w-x subpath of P, and Q', the w-y subpath of Q. Then $V(P') \cap V(Q') = \{w\}$.

But then the cycle formed by following P' from w to x, then the edge xy, and then following Q' in reverse from y to w is an odd cycle; more precisely, the cycle $w = v_j v_{j+1} v_{j+2} \ldots v_{2k-1} x y u_{2m-1} u_{2m-2} \ldots w$ has length $(2k - j) +)(2m - j) + 1 = 2(k + m - j) + 1$, which is odd.

But this contradicts the assumption that G has no odd cycles. Thus it must be that A is independent. A similar argument shows that B is independent.
So our result is proven.