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Hints and Solutions

Recall that
*If E is a finite extension of F and K is a finite extension of E, then $[K:F]=[K:E][E:F]$.

23. **Hint**: Use (*) above and the fact that $F \subseteq F(\alpha) \subseteq E$.

24. Solution: It is enough to show that $x^2 - 2$ has no zeros in $\mathbb{Q}\left(\sqrt[3]{2}\right)$. For this it is enough to show that $\sqrt{2} \notin \mathbb{Q}\left(\sqrt[3]{2}\right)$. However, if $\sqrt{2} \in \mathbb{Q}\left(\sqrt[3]{2}\right)$, then $2 = \deg\left(\sqrt{2}, \mathbb{Q}\right)$ would divide $3 = \deg\left(\sqrt[3]{2}, \mathbb{Q}\right)$.

26. **Hint**: Let $\alpha \in D$, $\alpha \neq 0$. It is enough to show that $\alpha^{-1} \in D$.
Since E is a finite extension of F, E is algebraic over F and so since $\alpha \in E$,
$F(\alpha) = \left\{a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n : a_i \in F\right\}$, where $n = \deg(\alpha, F)$.

29. **Hint**: Suppose that $\alpha \in E$ is a zero of $p(x)$. Then $\deg(p(x)) = [F(\alpha):F]$ - now use (*).

30. **Hint**: Since $F(\alpha)$ is a finite extension of F, it is an algebraic extension and so $\alpha^2 \in F(\alpha)$ must be algebraic over F. Suppose that $F(\alpha^2) \neq F(\alpha)$ and consider the value of $[F(\alpha):F(\alpha^2)]$.

31. You may use 31.11 in your text for this.