Name ___________________________

Math 547 – Practice Exam #3

1. (a). Explicitly describe the elements of the field $\mathbb{Q}(\pi)$.

   (b). Explicitly describe the elements of the field $\mathbb{Q}\left(\sqrt[3]{2}\right)$.

   (c). Give a basis for the field $\mathbb{Q}\left(\sqrt{3}, \sqrt{2}, i\right)$ over $\mathbb{Q}$.

   (d). Define algebraic extension.

2. **Prove**: If $F \subseteq K \subseteq E$ are fields and *K* is a finite extension of *F* and *E* is a finite extension of *K*, then $[E:F]=[E:K][K:F]$.

3. Suppose that γ is a zero of $p(x)=x^2+2x+3 \in Z_5[x]$ in some extension field *E*.
   Note: $p(x)=x^2+2x+3$ is irreducible in $Z_5[x]$; you need not verify this.

   (a). How many elements are there in $Z_5(\gamma)$? Explain.

   (b). Express the product $(1+2\gamma)(3+\gamma)$ in the form $a+b\gamma$, $a,b \in Z_5$.

   (c). Find an expression (in terms of γ) for the other zero of $p(x)=x^2+2x+3$ in *E*.

4. Let *D* be an integral domain with $F \subseteq D \subseteq E$ where *F* and *E* are fields and *E* is a finite extension of *F*. Show that *D* is a field.

5. Show directly that $\alpha=\sqrt{i+\sqrt{3}}$ is an algebraic number and determine its degree. Fully justify your answer.
   Hint: You may take as given that $\sqrt{i+\sqrt{3}} \notin \mathbb{Q}\left(i,\sqrt{3}\right)$.

6. Given that π is transcendental, show that $\sqrt{\pi}$ cannot be algebraic of degree at most 2.

7. Suppose that $p(x) \in F[x]$ is irreducible of degree *n* and that α is a zero of $p(x)$ in some extension field *E*. Thus $p(x)$ is the minimal polynomial for α.
   Let $S=\left\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\right\}$. Show that *S* is linearly independent in $F(\alpha)$.
   Note: Argue directly, you may not use the fact that *S* is a basis for $F(\alpha)$.

8. Let α be algebraic in *E* over *F* and suppose that $p(x)$ is its minimal polynomial. Then show that if $f(x) \in F[x]$ with $f(\alpha)=0$, then $p(x)$ divides $f(x)$.