Math 546
Problem Set 18

1. Prove: If G is Abelian, then every subgroup of G is normal.

   Solution: We noted this in class today.
   Proof. If H is a subgroup of the Abelian group G and $g \in G,\ h \in H$, then
   $ghg^{-1} = hgg^{-1} = he = h \in H$.

2. Prove: If H is a subgroup of G, then for any g in G, gHg^{-1} is also a subgroup of G.
   Solution: Note that $gHg^{-1} = \left\{ ghg^{-1} : h \in H \right\}$
   Clearly the identity is $e = geg^{-1} \in gHg^{-1}$.
   If x and y belong to gHg^{-1}, then $x = gh_1g^{-1},\ \ y = gh_2g^{-1}$ for some elements
   $h_1, h_2 \in H$. Thus, $xy = gh_1g^{-1}gh_2g^{-1} = gh_1h_2g^{-1} \in gHg^{-1}$. So gHg^{-1} is closed.
   Finally if $x = ghg^{-1} \in H$, then $x^{-1} = (ghg^{-1})^{-1} = gh^{-1}g^{-1} \in H$ (since $h^{-1} \in H$).

3. Prove the theorem below.
   Theorem. Let G be a group and H a subgroup of G. Then the following are equivalent:
   (i). $H \triangleleft G$.
   (ii). For every g in G, $gH = Hg$.
   (iii). For every g in G, $gHg^{-1} = H$.
   Solution: $(i) \Rightarrow (ii)$. Suppose that $H \triangleleft G$ and let x be any element of gH.
   Then $x = gh$ for some h in H. Thus, $h_1 = ghg^{-1} \in H$. And so,
   $x = gh = ghg^{-1}g = h_1g \in Hg$. Thus we have shown that $gH \subseteq Hg$, Essentially the same argument shows that $Hg \subseteq gH$ and hence $gH = Hg$.
   Now show (ii) implies (iii) and (iii) implies (i).

4. (a). If H is a subgroup of the group G and $[G : H] = 2$, then $H \triangleleft G$.
   Hint: Consider problem 3(ii).
   Solution: Suppose that g is any element of G. If g is in H, then $gH = H = Hg$.
   If g does not belong to H, then gH is the left coset that is different from H and Hg is the right coset that is different from H and so $gH = Hg$.

   (b). Show that $A_4 \triangleleft S_4$.

5. A_4 has exactly one subgroup of order 4, namely
$K = \{i,\ (1,2)(3,4),\ (1,3)(2,4),\ (1,4)(1,3)\}$.
Show that K is a normal subgroup of A_4.
Hint: Refer to problems 2 & 3.
Solution: Since K is the only subgroup of order 4 and since gKg^{-1} is a subgroup of order 4, then $gKg^{-1} = K$ and so K is normal by problem 3.

6. Recall that $SL(2,R) = \{A \in GL(2,R) : \det(A) = 1\}$. Show that $SL(2,R) \lhd GL(2,R)$.
Hint: If A and B are both $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.