1. **Prove:** If $A$, $B$ and $C$ are sets and $f : A \to B$, $g : B \to C$ are both onto, then so is $g \circ f : A \to C$.

**Solution:**
In order to show that $f$ is onto, we must show that for every element $c$ of $C$, $g \circ f (a) = c$ for some $a$ in $A$.

However, since $g$ is onto, there is some $b$ in $B$ such that $g(b) = c$ and since $f$ is onto, there is some $a$ in $A$ such that $f(a) = b$. Hence, $g \circ f (a) = g(f(a)) = g(b) = c$.

2. **Prove:** If $A$, $B$ and $C$ are sets and $f : A \to B$, $g : B \to C$ are both 1-1, then so is $g \circ f : A \to C$.

**Solution:**
Suppose that $x$ and $y$ are different elements of $A$. we need to show that $g \circ f (x) \neq g \circ f (y)$.

However, since $f$ is 1-1, $x \neq y \Rightarrow f(x) \neq f(y)$.

And then since $g$ is 1-1, $f(x) \neq f(y) \Rightarrow g(f(x)) \neq g(f(y)) \Leftrightarrow g \circ f (x) \neq g \circ f (y)$.

3. **Suppose that $A$ is a non-empty set and $F_A$ is the set of all functions from $A$ to $A$.**
   (a) If $|A| = n$, then how many elements are there in $F_A$?

   (b) Convince yourself that each of the following subsets of $F_A$ is a subsemigroup under the composition of functions.
   
   (i). The set of onto functions from $A$ to $A$.
   
   (ii). The set of 1-1 functions from $A$ to $A$

   (iii). The set of bijective functions from $A$ to $A$.

   **Solution:** By problem (1) with $B$ and $C$ both replaced by $A$, the set of onto functions from $A$ to $A$ is closed under composition. So, since we know that $F_A$ is a semigroup, the set of onto functions is also a semigroup.

4. Show that if $A$ is a finite non-empty set, then for any function $f : A \to A$,
   (a) If $f$ is onto, then $f$ is also 1-1.

   **Solution:**
   Assume that $A$ is a finite non-empty set, and that $f : A \to A$ is onto.
   Suppose that $f$ is not 1-1. Then there exists $a, b \in A$ such that $a \neq b$, but $f(a) = f(b)$.
   Now let $B = A - \{a\}$. Then the function $f_1 : B \to A$ defined by $f_1(x) = f(x)$ for
all \( x \) in \( B \) is also onto \( A \). But this means that \(| A | \leq | B |\) which is impossible since \( B \) is a smaller set than \( A \).

(b). If \( f \) is 1-1, then \( f \) is also onto.

**Solution:**
Suppose that \( f \) is 1-1, but not onto. Then since \( f \) is not onto, there is some element \( a \) in \( A \) such that \( f(x) \neq a \) for every element \( x \) of \( A \).
But now let \( B = A - \{a\} \). Then the function \( f_1 : A \rightarrow B \) defined by \( f_1(x) = f(x) \)
for all \( x \) in \( A \) is 1-1. But this means that \(| A | \leq | B |\) which is impossible since \( B \) is a smaller set than \( A \).

(c). Show that neither of (a) nor (b) is true if \( A \) is infinite.

**Solution:**
To see that (b) need not be true when \( A \) is infinite, choose \( A = \mathbb{Z}^+ \), and let \( f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \) be defined by \( f(n) = n + 1 \). Then clearly \( f \) is 1-1, but it is not onto (why?)

For part (a), can you find an example of a function \( f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \) that is onto but not 1-1?

5. Let \( A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \) and define \( f:A \rightarrow P(A) \) by the following assignments:

\[
\begin{align*}
1 & \rightarrow \{2, 3, 4\}, \quad 2 \rightarrow \{1, 2, 3\}, \quad 3 \rightarrow \{1, 6, 9\} \quad 4 \rightarrow \emptyset \\
5 & \rightarrow \{5\}, \quad 6 \rightarrow \{2, 6, 9\}, \quad 7 \rightarrow \{2, 8\} \quad 8 \rightarrow \{1, 9\}
\end{align*}
\]

Then determine the set \( M \) in the proof of Cantor’s Theorem as applied to \( A \).

6. Suppose that \((S, *)\) is a binary system and suppose that for any \( a \) and \( b \) in \( S \), \( a * (b * a) = b \).
Show that for any \( a \) and \( b \) in \( S \), \((a * b) * a = b \). (This is harder than the others.)

7. Let \( A = \{a, b, c\} \). There are six bijections from \( A \) to \( A \).
Note: bijections on a finite set are also often called permutations.
Let’s give these functions some names: \( i, x, y, z, r \) and \( s \) defined as follows:

\[
\begin{align*}
i(a) & = a, \quad i(b) = b, \quad i(c) = c \\
x(a) & = a, \quad x(b) = c, \quad x(c) = b \\
y(a) & = b, \quad y(b) = a, \quad y(c) = c \\
z(a) & = c, \quad z(b) = b, \quad z(c) = a \\
r(a) & = b, \quad r(b) = c, \quad r(c) = a
\end{align*}
\]
\[ s(a) = c, \ s(b) = a, \ s(c) = b \]

Complete the following ‘multiplication table’ for the semigroup of these bijections under composition.

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**Solution:**