Math 546
Problem Set 3

1. For the semigroup $(R^+, *)$ where $a * b = \dfrac{ab}{a+b}$, let x be a positive real number, and express x^4 (where the power is in the semigroup, not the ordinary power) as a simple expression in terms of x.

   If you like, you may write $x^{\underline{n}}$ to distinguish the n^{th} power of x in the semigroup from the ordinary power of x.

   Solution: $x^{\underline{4}} = \dfrac{x}{4}$

   $x^{\underline{2}} = x * x = \dfrac{x^2}{2x} = \dfrac{x}{2}$. $x^{\underline{3}} = x * x^{\underline{2}} = \dfrac{x\left(\frac{x}{2}\right)}{x + \frac{x}{2}} = \dfrac{x}{3}$.

   $x^{\underline{4}} = x * x^{\underline{3}} = \dfrac{x\left(\frac{x}{3}\right)}{x + \frac{x}{3}} = \dfrac{x}{4}$. I bet you see a pattern here!

2. Why isn't $(R, *)$ where $a * b = \dfrac{ab}{a+b}$ a semigroup?

   Note: it *is* a semigroup if R is replaced by R^+ (see problem #1).

   Solution: It is not even a binary system because (for example), $3 * (-3)$ is undefined.

3. For the semigroup $(R, *)$ where $a * b = a + b - ab$, what is the value of
   (a). $2^{\underline{3}}$ (b). $3^{\underline{4}}$

   Solution: (a). 2 (b). –15

4. Suppose that $(S, \cdot)$ is a semigroup and that x is an element of S such that $x^{12} = x$. Then there is some positive integer m such that x^m is an idempotent. What is that value of m?
   Answer: $m = 11$

5. Finish the proof of the theorem that every finite semigroup has an idempotent.

6. Let $(S,\cdot)$ be a commutative binary structure.
   Show that for all $a, b \in S$, $a \cdot (b \cdot a) = (a \cdot b) \cdot a$

7. **Prove**: $\log_3 2$ is not a rational number.
   Solution:

   Suppose that $\log_3 2$ is rational. Then $\log_3 2 = \frac{a}{b}$ for some integers a and b.

   But …FINISH.

8. Show that there exist irrational numbers a and b such that a^b is rational. There should not be any gaps in your argument. So for this exercise, you may not assume that e or ln 2 are irrational unless you can prove it.

   Hints: Here is a (fiendishly clever) suggestion for one way to settle the problem.
   1. Consider the number $\sqrt{2}^{\sqrt{2}}$
   2. You probably can't tell whether this number is rational or not.
   3. Consider the cases of whether it is or is not rational.

9. Can the table below be completed to that of a semigroup?
   Justify your answer. You may refer to anything we have stated as true in class.

*	a	b	c	d
a	b			
b		c		
c			d	
d				a

10. Suppose that $(S, *)$ is a semigroup and let a, b be two particular elements of S.
    Given that
    (i). $a * b = b * a$
    (ii). There is *exactly one* element t in S such that $a * t = t$.
    Show that $b * t = t$.