Lecture 08
13.1/13.2/13.3: Smooth curves, integrals, arc length

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Things to note

Exam 1 is on Monday, February 11 (1 week).

Quiz 04 will be on Wednesday, February 6 (next class).

Friday, February 8 will be a review day with no quiz.

MTW office hours canceled.

Quiz 03 Key ideas

1. *Intersection* (a geometric idea) means *substitution* algebraically.

$$2(1+t)-(-2+5t)+3(3-2t)=40$$

2. A plane being perpendicular to a line means the plane's normal vector is parallel to the direction vector of the line.

$$\vec{\mathbf{n}}=\langle -2,3,-1\rangle$$

Last class

Definition

A vector-valued function is a function

$$\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t) \rangle$$

where f, g, and h are real-valued functions.

Definition

Let $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t) \rangle$. $\vec{\mathbf{r}}$ is differentiable at $t = t_0$ if f, g and h are differentiable at t_0. In this case,

$$\vec{r}'(t) = \frac{d\vec{\mathbf{r}}}{dt} = \left\langle \frac{df}{dt}, \frac{dg}{dt}, \frac{dh}{dt} \right\rangle.$$

$$\vec{\mathbf{v}}(t) = \frac{d\vec{\mathbf{r}}}{dt} \text{ and } \vec{\mathbf{a}}(t) = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d^2\vec{\mathbf{r}}}{dt^2}.$$

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1. *$\frac{d\vec{\mathbf{r}}}{dt}$ is continuous on D, and*
2. *$\frac{d\vec{\mathbf{r}}}{dt}$ is never $\vec{\mathbf{0}}$ on D.*

We simply say $\vec{\mathbf{r}}(t)$ is smooth if $\vec{\mathbf{r}}(t)$ is smooth on $\mathbb{R}$.

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Show that the helix $\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t), t \rangle$ *is smooth.*

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Example

Show that the helix $\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t), t\rangle$ is smooth.

We have $\frac{d\vec{\mathbf{r}}}{dt} = \langle -\sin(t), \cos(t), 1\rangle$. Since the z-direction of $\frac{d\vec{\mathbf{r}}}{dt}$ is always 1, $\frac{d\vec{\mathbf{r}}}{dt}$ is never the zero vector for any value of t. Thus $\vec{\mathbf{r}}(t)$ is smooth.

Vector Functions of Constant Length

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If $\vec{\mathbf{r}}(t)$ has constant length, then we have $\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}(t) = c^2$ for some length c. This means

$$\frac{d}{dt}\left[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}(t)\right] = 0 \Rightarrow \frac{d\vec{\mathbf{r}}}{dt} \cdot \vec{\mathbf{r}}(t) + \vec{\mathbf{r}}(t) \cdot \frac{d\vec{\mathbf{r}}}{dt} = 0 \Rightarrow 2\vec{\mathbf{r}}(t)\frac{d\vec{\mathbf{r}}}{dt} = 0$$

which gives the desired result.

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$$\int \vec{\mathbf{r}}(t)dt = \overrightarrow{\mathbf{R}}(t) + \vec{\mathbf{C}}$$

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Example

Let $\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t), t \rangle$*. Find* $\int \vec{\mathbf{r}}(t)dt$*.*

The integral is $\langle \sin(t), -\cos(t), \frac{t^2}{2} \rangle + \langle c_1, c_2, c_3 \rangle$ where the c_i are constant real numbers.

Definite Integrals

Definition

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$$\int_a^b \vec{\mathbf{r}}(t)dt = \left\langle \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt \right\rangle$$

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Example

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$= [\sin(t)]_0^{\pi}\vec{\mathbf{i}} + [t]_0^{\pi}\vec{\mathbf{j}} + [t^2]_0^{\pi}\vec{\mathbf{k}} = (0-0)\vec{\mathbf{i}} + (\pi - 0)\vec{\mathbf{k}} + [\pi^2 - 0]\vec{\mathbf{k}} = \langle 0, \pi, \pi^2\rangle$.

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Recall that the arc length of a parametrized curve $x = f(t)$, $y = g(t)$ from $t = a$ to $t = b$ is given by the formula

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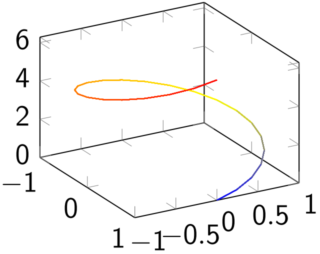
Definition

Let $\vec{\mathbf{r}}(t) = f(t)\vec{\mathbf{i}} + g(t)\vec{\mathbf{j}} + h(t)\vec{\mathbf{k}}$ be smooth and let $a \leq t \leq b$. Then the length of $\vec{\mathbf{r}}$ from $t = a$ to $t = b$ is

$$L = \int_a^b \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}\, dt.$$

Example

Find the length of the helix $\vec{\mathbf{r}}(t) = \langle \cos(t), \sin(t), t \rangle$ *from* $t = 0$ *to* $t = 2\pi$.



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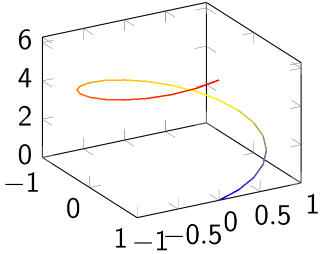

The length is

$$L = \int_0^{2\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1^2}\,dt = \int_0^{2\pi} \sqrt{1+1}\,dt$$

$$= \sqrt{2}t \Big]_{t=0}^{t=2\pi} = 2\sqrt{2}\pi.$$