Math 758S

Lecture for 4/27/00

Review of General Theory and Motivation

To wrap up this portion of the course, we briefly review what we have proved concerning unconditional basis representations and how it relates to multiresolution and approximation in classical spaces. For $L^2(\mathcal{T})$ functions, we have proved (Lecture 35) that Fourier series are almost everywhere convergent for almost all choices of signs, however, until Carleson proved his theorem (each L^2 function has almost everywhere convergent Fourier series) it was not known whether the choice of signs with all pluses was one such series. We then proved (Lectures 36-37) that the classical Fourier series of L^p functions ($p \neq 2$) do not give rise to unconditional bases and in fact for almost every choice of signs we have almost every divergence (with respect to any summability method) of the resulting series. Together with the Hunt's extension of the Carleson theorem to L^p, $1 < p < \infty$, this shows although the Fourier series of an L^p function converges almost everywhere,that almost all rearrangements and changes of signs for the classical Fourier series of functions which are not square integrable cannot be the Fourier series of any other reasonable function.

The failure of Fourier series in this respect provides additional importance to wavelet representations since their unconditional representations in classical spaces provides the basis for the success of nonlinear approximation methods, i.e., as we have previously described, being able to select wavelet components in an arbitrary order depending upon weights determined by the approximation problem being considered. First we restate the general results we have developed for unconditional bases.

Theorem. (Lectures 31-32) *Suppose $\mathcal{B} = \{f_j, \lambda_j\}_{j\in\mathcal{N}}$ is a basis for a Banach space B (although an abuse of notation, we denote the basis in short by $\{f_j\}$, then the following conditions are equivalent:*

(a) $\{f_j, \lambda_j\}_j$ *is an unconditional basis*

(b) $\{f_{\pi(j)}\}_j$ *is a basis for each permutation π of the integers.*

(c) *for each $f \in B$, and every bounded sequence of multipliers $|\beta_j| \leq 1$ the series $\sum_j \beta_j \lambda_j(f) f_j$ converges in B.*

(d) *for each $f \in B$, and every choice of signs $\epsilon_j = \pm 1$, the series $\sum_j \epsilon_j \lambda_j(f) f_j$ converges in B.*

Moveover, we proved by Baire category arguments that for unconditional bases, the corresponding operators of partial sums of the expressions in conditions **(b)-(d)** *are uniformly bounded.*

Unconditional Bases in L^p

In the particular case that the spaces are L^p spaces, we first showed that the Haar basis is unconditional for L^p, $1 < p < \infty$ (Lecture 33). The proof used the Calderon-Zygmund decomposition to establish a weak type (1,1) and standard interpolation methods for the intermediate estimates.

Defn. The corresponding Paley square function for a biorthogonal basis $\mathcal{B} = \{f_j, \lambda_j\}_{j\in\mathcal{N}}$ is defined as the quadratic functional

$$\mathcal{S}_{\mathcal{B}}[f](x) := \left(\sum_j |\lambda_j(f) f_j(x)|^2 \right)^{\frac{1}{2}}.$$

In the case of a wavelet basis a somewhat cruder version of Paley's function is defined by

$$\mathcal{P}[f](x) := \left(\sum_{I\in\mathcal{D}} |c_I(f)\chi_I(x)|^2 \right)^{\frac{1}{2}}$$

when f has the representation $f = \sum_I c_I(f)\psi_I$. Note that $\mathcal{P}[f] = \mathcal{S}_{\mathcal{H}}[F]$ where $\mathcal{H}$ is the Haar basis and $F := \sum_I c_I(f) H_I$.

Using Khinchin's inequality (see Lecture 27), we were able to provide a further characterization of unconditional biorthogonal bases.

Theorem. (see Lecture 34) *A necessary and sufficient condition for $\mathcal{B}$ to be unconditional for L^p, $1 < p < \infty$ is that its corresponding square function satisfy*

$$c_1 \|f\|_p \leq \|\mathcal{S}_{\mathcal{B}}[f]\|_p \leq c_2 \|f\|_p \tag{1}$$

for some positive constants c_1, c_2 independent of $f \in L^p$.

The Fefferman-Stein Vector-valued Maximal Inequality

Fefferman and Stein [2] considered the maximal function $\mathbf{F} = \mathbf{M}(\mathbf{f})$ where $\mathbf{F}_j := M(f_j)$ and f_j denotes the j-th component of $\mathbf{f}$. A corresponding quadratic scalar-valued maximal function

$$\overline{M}\mathbf{f}(x) := \|\mathbf{M}\mathbf{f}(x)\|_{\ell^2}$$

was used in their development of real methods for Hardy spaces [3], in particular to analyze Littlewood-Paley decompositions and and the study of the *grand maximal operator*.

In [2], they proved for $1 < p < \infty$ the mixed-norm estimate

$$\|\mathbf{M}\mathbf{f}\|_{L^p(\ell^2)} \leq c \, \|\mathbf{f}\|_{L^p(\ell^2)}$$

or equivalently in Stein's notation

$$\|\overline{M}\mathbf{f}\|_{L^p} \leq c \, \|\mathbf{f}\|_{L^p(\ell^2)}. \tag{2}$$

which is the same as

$$\left\| \left(\sum_j |M(f_j)|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \leq c \left\| \left(\sum_j |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p}.$$

This is obvious (Fubini's theorem) when $p = 2$. For the case of $1 < p < 2$, the proof uses the now familiar arguments: weak-type (1,1) inequality and Marcinkiewicz interpolation. The weak type inequality uses a Calderon-Zygmund decomposition of $\mathbf{f}$ where the averages are Bochner integrals (as we saw last semester for vector-valued singular integral operators). A Xerox copy of the proof from Stein's book has been placed in your mailboxes, but with corrections. Namely, *delete* the first sentence at the top of page 53, which reads "Write $b_j = b\chi_{Q_j}$, $b_j^o = B^o\chi_{Q_j}$, so $b = \sum b_j$ and $b^o = \sum b_j^o$." After the next sentence, *insert* "where b_j is the j-th component of $\mathbf{b}$." In the displayed equation on line 8, *replace* the two instances of "$\sum\limits_k$" by "$\sum\limits_{Q_k \cap B \neq \emptyset}$", i.e., the sum over all the cubes of the Calderon-Zygmund decomposition which intersect B.

The proof for the case $2 < p < \infty$ introduces a new idea - *weighted norm inequalities* - and the key is a sort of nonlinear adjoint estimate for the scalar-valued Hardy-Littlewood maximal operator:

Proposition. For non-negative locally integrable *weight* w, define the measure $w(E) := \int_E w(x)dx$, then we have

$$w\{Mf(x) > \alpha\} \leq \frac{C}{\alpha} \int_{R^d} |f(x)| \, Mw(x) \, dx$$

and for $1 < q < \infty$

$$\int_{R^d} |Mf(x)|^q \, w(x) \, dx \leq C_q \int_{R^d} |f(x)|^q \, Mw(x) \, dx$$

We mention that one may simplify the proof given there if one uses our standard maximal operator

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy$$

instead of the centered maximal function used in Stein's text

$$M_c f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy.$$

As we saw last semester, it is easy to verify that these two operators are, to within constants, pointwise equivalent, but in many cases technicalities can be avoided by using the more general operator in the former expression. This is one such case as the proof given on page 54 can then be reduced to the key estimate

$$\int_{3Q_k} w \, dx \leq 3^d \, |Q_k| \inf_{y \in 3 \cdot Q_k} Mw(y) \leq \frac{C}{\alpha} \int_{Q_k} |f(y)| Mw(y) \, dy$$

which follows immediately from the Calderon-Zygmund decomposition for f, since

$$\alpha < \frac{1}{|Q_k|} \int_{Q_K} |f(y)| \, dy.$$

Upon summing over k, we get the desired weak type estimate since $\cup_k (3 \cdot Q_k)$ contains E_α and the Q_k's are disjoint. One may then continue with the standard duality argument beginning in the middle of page 55.

Classical Spaces and Wavelet Representations

Following the development of DeVore, Konyagin, and Temlyakov [1], we can now summarize the proof of wavelet characterizations of the functions in Lebesgue spaces L^p for $1 < p < \infty$. We begin with a ψ that provides a multiresolution analysis for L^2 (see Lecture 38). Under extremely mild conditions on ψ, namely

$$|\psi(x)| \leq M\chi_{[0,1]}(x), \quad a.e.$$

$$\chi_{[0,1]}(x) \leq M\psi(x), \quad a.e.,$$

it follows from the vector-valued estimates for the Hardy-Littlewood maximal operator that the corresponding Paley square functions for the wavelet basis $\Psi = \{\psi_I, \tilde{\psi}_I\}_{I\in\mathcal{D}}$ and the Haar basis H are norm-equivalent

$$\|\mathcal{S}_\Psi[f]\|_p \approx \|\mathcal{S}_\mathcal{H}[F]\|_p \approx \|\mathcal{P}[F]\|_p \tag{3}$$

where again $F := \sum_I c_I(f)H_I$ if $f = \sum_I c_I(f)\psi_I$. Indeed, by dilations and translations, if I is any dyadic interval, then

$$|\psi_I| \leq M\chi_I, \quad a.e.$$

$$\chi_I \leq M\psi_I, \quad a.e.$$

and the estimate (3) follows immediately from the Fefferman-Stein inequality (2).

The final step to prove that the wavelet bases are unconditional (in L^p) is to use the characterization provided by condition (1). To establish this we just need to prove under certain conditions on ψ (see condition (6) below) that the mapping

$$T\left(\sum_{I\in\mathcal{F}} c_I H_I\right) := \sum_{I\in\mathcal{F}} c_I\psi_I$$

is bounded on L^p

$$\| \sum_{I\in\mathcal{F}} c_I\psi_I\|_{L^p} \leq c \,\| \sum_{I\in\mathcal{F}} c_I H_I\|_{L^p} \ . \tag{4}$$

with constant c independent of the finite set $\mathcal{F}$. If this holds, then using the adjoint operator,

$$T^*(\sum_{I\in\mathcal{F}} c_I\psi_I) := \sum_{I\in\mathcal{F}} c_I H_I$$

the inverse estimate is also valid, namely

$$\| \sum_{I\in\mathcal{F}} c_I H_I\|_{L^p} \leq c \,\| \sum_{I\in\mathcal{F}} c_I\psi_I\|_{L^p} \ . \tag{5}$$

Together these estimates ((4), (5), and (3)) show that equivalence condition (1) is verified and therefore the basis is unconditional. To prove inequality (4), the basis $\mathcal{B}$ is expanded in terms of the Haar basis $\mathcal{H}$ in order to estimate the norm of the associated ("almost diagonal") transformation matrix $A = (a(I,J))_{I,J}$ with entries

$$a(I,J) := \int \psi_J H_I \, dx \ .$$

The operator T can be shown to be bounded if the corresponding matrix entries satisfy a decay condition away from the diagonal

$$|a(I,J)| \le C\, w(I,J) \tag{6}$$

where for some $\epsilon > 0$

$$w(I,J) := \left(1 + \frac{|x_I - x_J|}{\max(|I|,|J|)}\right)^{-1-\epsilon} \min\left(\frac{|I|}{|J|}, \frac{|J|}{|I|}\right)^{\frac{1+\epsilon}{2}}$$

and x_I denotes the center of I. Once again, the Fefferman-Stein inequality comes into play here in a decisive way, along with the previously established fact that the Haar basis is unconditional. See Theorems 4.1-4.2 of [1] for the technical details.

References

[1] R.A. DeVore, S.V. Konyagin, and V.N. Temlyakov, "Hyperbolic wavelet approximation," *Constr. Approx.* **14** (1998), 1–26.

[2] C. Fefferman and E.M. Stein,"Some maximal inequalities," *Amer. J. Math.* **93**, 107–115.

[3] C. Fefferman and E.M. Stein, "H^p spaces of several variables," *Acta Math.* **129**, p. 127-193.