Math 750 - Introduction

Outline of Fourier's method

We show that under certain assumptions that heat is governed by a partial differential equation on a domain D^*

$$\frac{\partial u}{\partial t} = \kappa \, \Delta u \tag{1}$$

and satisfies the initial-boundary value conditions

$$\begin{aligned} u(x,0) &= f(x), \qquad && x \in D^* \cup \partial D^* \\ u(x,t) &= g(x,t), \qquad && x \in \partial D^*, t > 0. \end{aligned} \tag{2}$$

Here $u(x,t)$ is the temperature of the body at location $x \in D^*$ at time t. Fourier formulated the empirical law for heat flow (i.e. Fourier's law)

$$\mathbf{q}(x,t) = -K \, \nabla u \tag{3}$$

which states that the time rate of change of heat is proportional to the spatial rate of change of the temperature (with heat 'flowing' from warmer regions to cooler regions). K is called the *thermal conductivity* and depends upon the material properties of the body D^*. For convenience we assume that K is constant.

Rate of heat passing through ∂D. The amount of heat *leaving* a small volume $D \subset D^*$ through its boundary is equal to the surface integral

$$\int_{\partial D} \mathbf{q} \cdot \mathbf{n} \, dA \tag{4}$$

Applying the divergence theorem to this equation, we get

$$\int_{\partial D} \mathbf{q} \cdot \mathbf{n} \, dA = \int_D \nabla \cdot \mathbf{q} \, dV \tag{5}$$

and so substituting for $\mathbf{q}$ from Fourier's law (3), we obtain that the heat flow the small volume D ***gains*** (hence the change in sign) is equal to

$$\int_D K \Delta u \, dV \tag{6}$$

This is under the assumption that there are no sources or sinks of heat inside D.

Rate of change of heat of D. On the other hand the total heat in the small volume D is equal to the volume integral

$$\int_D \sigma \rho u \, dV \tag{7}$$

where σ is the material *specific heat* (heat gain per unit mass per unit change in temperature) and ρ is the material *mass density* (mass per unit volume). Therefore the time rate of change of the heat equals

$$\frac{d}{dt}\left(\int_D \sigma \rho u \, dV\right) = \int_D \sigma \rho \, \frac{\partial u}{\partial t} \, dV \tag{8}$$

where in this equation we have used Leibnitz's rule for differentiation of integrals and for convenience have assumed that σ and ρ are constant.

Heat Balance. We equate the two rates in expressions (6) and (8) (i.e. the heat balance) subtract, and divide by the volume of D to obtain

$$\frac{1}{\text{vol.}(D)} \int_D \left(\sigma\rho \, \frac{\partial u}{\partial t} - K\Delta u) \right) \, dV = 0. \tag{9}$$

But D is an arbitrary region inside D^*, so we may vary D about any point in D^* letting its volume tend to zero in (9), and apply the (Lebesgue) Differentiation Theorem to formally obtain that

$$\frac{\partial u}{\partial t} - \kappa \, \Delta u = 0, \quad a.e. \tag{10}$$

where $\kappa := \frac{K}{\sigma\rho}$. If these quantities are continuous, then the PDE for u holds on all of D^*.

Solution in 1 D. We use the elementary separation of variable technique: assume a solution of the form

$$U(x,t) = X(x) \, T(t), \tag{11}$$

substitute into the PDE, divide by U to obtain

$$\frac{T'}{T} = \kappa \, \frac{X''}{X}. \tag{12}$$

The left hand side is only a function of t, while the right hand side is only a function of x, so they are both constant and coupled through this relationship:

$$\frac{X''}{X} = -\lambda \tag{13}$$

and

$$\frac{T'}{T} = -\lambda\kappa. \tag{14}$$

For illustration, we take the 1-D domain D^* to be the interval $[0,\pi]$ and take zero boundary conditions in (2). By considering all possible cases in (15), we see that the eigenvalues $0 < \lambda = n^2, (n \in I\!N)$ is necessary for any nonzero solution to the PDE, and

$$X_n(x) = b_n \sin(nx) \tag{15}$$

(If we take the domain as $[-\pi/2, \pi/2]$ for even functions f, then $X_n(x) = a_n \cos(nx)$). Substituting this value of λ into the coupled equation for the temporal component, we get

$$T_n(t) = \exp(-n^2\kappa t) \tag{16}$$

and so the tensor product solution is

$$U_n(x,t) = b_n \sin(nx) \, \exp(-n^2\kappa t). \tag{17}$$

Superimposing the solutions U_n and using the linearity of the initial-boundary value problem, we see that a solution is of the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp(-n^2\kappa t) \, \sin(nx). \tag{18}$$

Setting $t = 0$, a formal solution is found by solving for coefficients $\{b_n\}$ in the representation:

$$f(x) = \sum_{n=1}^{\infty} b_n \, \sin(nx). \tag{19}$$

Now the analysis takes center stage: When and in what sense does this sum converge, that is when does *equality* hold and for what f is the representation (19) valid? Once this is addressed, when/where does the solution (18) converge and solve the PDE, i.e. is this really a solution in D^* when it does converge? Is this solution unique? As $t \downarrow 0$, in what sense does $u(\cdot,t)$ converge to f?