Math 750 - HW # 1

Due Thursday - Sept. 7

1. Problems on Cesaro means:

   (a) Turn in (after all) the earlier assigned problem on Cesaro means of sequences: Convergence implies Cesaro summability.

   (b) Suppose that both $\{a_n\}_n$ and $\{b_n\}_n$ are Cesaro summable, then show that the sequence $c_n := \alpha a_n + \beta b_n$ is also.

   (c) Determine the Cesaro means of the sequence $\left\{(-1)^n - \dfrac{1}{n}\right\}$.

2. How can you tell if a norm actually arises from an inner product? Use a 'bullet' to show that the inner product in an inner product space over the real scalars can be computed from the norm by the formula:

$$< f, g > \; = \; \frac{1}{2}\left(\|f\|^2 + \|g\|^2 - \|f - g\|^2\right)$$

   Extra Credit: Derive a similar formula when the scalar field is $\mathcal{C}$.

3. Consider the collection $\phi_n(t) = \cos(nt)$ in $L^2(-\pi, \pi)$. Show that this collection is orthogonal where the inner product is given by $\int_{-\pi}^{\pi} f(t)\overline{g(t)}dt$. What is the norm of ϕ_n? (Hint: use the trig identity $\cos(a+b) + \cos(a-b) = 2\cos(a)\cos(b)$.)

4. A metric space is called *separable* if it has a countable dense subset. Let H be a Hilbert space with the natural metric (i.e. $d(f, g) := \|f - g\|_H$) and let $\Phi := \{\phi_\alpha\}_\alpha$ be any orthonormal collection from H.

   (a) Compute the distance between any two distinct members of Φ.

   (b) Prove that Φ must be countable if H is separable.

   (c) Sketch the proof that if Φ is countable and maximal in the partial ordering of set inclusion (existence by Zorn's lemma), then H is separable.

5. Let X be a Banach space (i.e. a complete normed linear space). Prove that if $M_n := \|f_n\|_X$ and the sequence $\{\sum_{n=1}^N M_n\}_N$ is bounded in $\mathcal{R}$, then the series $\sum_{n=1}^\infty f_n$ converges in X. (Recall that for the series converge, we just mean that the sequence of partial sums convergence as a sequence in X).