As we shall see, the continuity of the function is not sufficient to guarantee that the interpolation polynomial actually approximate the function as the number of nodes increases. In the following chapter we will support this assertion with appropriate negative results. If we impose additional conditions on the function $f(x)$, we obtain positive results when the increase in the number of nodal points proceeds according to a suitable rule.

The very same problems arise in interpolating with trigonometric polynomials; this manner of interpolating is more natural when the function to be approximated is 2π-periodic.

§2. The Lagrange Formula

Let us consider the following problem: let two sets each consisting of n real numbers

$$x_1, x_2, x_3, \dots, x_n, \tag{1}$$

$$y_1, y_2, y_3, \dots, y_n, \tag{2}$$

be given where the numbers (1) are pairwise distinct (we do not require this of the numbers (2)). It is required to find a polynomial of lowest possible degree satisfying the equations

$$L(x_i) = y_i \qquad (i = 1, 2, \dots, n). \tag{3}$$

To solve the problem it is sufficient to note that for the polynomial

$$l_k(x) = \frac{(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)} \tag{4}$$

the equations

$$l_k(x_i) = \begin{cases} 0 & \text{for } i \neq k, \\ 1 & \text{for } i = k \end{cases}$$

hold.

Condition (3) is therefore satisfied by the polynomial

$$L(x) = \sum_{k=1}^{n} y_k l_k(x). \tag{5}$$

The degree of this polynomial is at most $n - 1$. There is however no other polynomial $M(x)$ from H_{n-1} which satisfies condition (3); if this were the case, then the difference $L(x) - M(x)$ would be a polynomial of H_{n-1} which would not be identically zero and would possess the n roots (1), but this is impossible. The polynomial $L(x)$ is hence the unique solution of the problem. Formula (5), which represents this polynomial in terms of the x_i and y_i, is called the Lagrange *interpolation formula*.

The polynomial $l_k(x)$, which is called a *basis polynomial*, can be written in a more abbreviated form. If we define

$$\omega(x) = (x - x_1)(x - x_2)\dots(x - x_n), \tag{6}$$

then

$$(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n) = \frac{\omega(x)}{x - x_k},$$

$$(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n) = \lim_{x \to x_k} \frac{\omega(x)}{x - x_k} = \omega'(x_k),$$

whence

$$l_k(x) = \frac{\omega(x)}{\omega'(x_k)(x - x_k)}. \tag{7}$$

If $P(x)$ is any polynomial of H_{n-1} and $x_1, x_2, \dots, x_n$ are distinct values of its argument, then the equation

$$P(x) = \sum_{k=1}^{n} P(x_k) l_k(x), \tag{8}$$

holds, since both sides of this equation are polynomials from H_{n-1} which coincide at the n points x_i. In particular,

$$\sum_{k=1}^{n} l_k(x) = 1. \tag{9}$$

Now if $f(x)$ is an arbitrary function defined on the interval $[a, b]$ and the x_i are particular nodal points chosen in this interval, then

$$L(x) = \sum_{k=1}^{n} f(x_k) l_k(x) \tag{10}$$

is the unique polynomial of H_{n-1} which coincides with $f(x)$ at the nodes x_i. Of course, $L(x)$ and $f(x)$ may differ at all points $x \neq x_i$. The polynomial (10) is called the Lagrange *interpolation polynomial for the function* $f(x)$. In order to emphasize its dependence on the function, we sometimes denote it by $L[f; x]$. Formula (8) then implies

$$L[P; x] = P(x), \tag{11}$$

if $P(x)$ is a polynomial of H_{n-1}.

Now suppose that on its domain of definition $[a, b]$ the function $f(x)$ possesses a finite derivative of nth order. In this case we are able to construct a useful expression for the difference $f(x) - L(x)$ at the non-nodal points. For such a point x (which is now to be considered fixed in the interval $[a, b]$) we define[2]:

[2] Since $x \neq x_i$ $(i = 1, 2, \dots, n)$, $\omega(x) \neq 0$.

$$K = \frac{f(x) - L(x)}{\omega(x)} \tag{12}$$

and

$$\varphi(z) = f(z) - L(z) - K\omega(z).$$

This function is defined on $[a, b]$ and possesses a finite nth derivative

$$\varphi^{(n)}(z) = f^{(n)}(z) - Kn!, \tag{13}$$

since $L(z)$ is a polynomial from H_{n-1}, and $\omega^{(n)}(z) = n!$. It is obvious that

$$\varphi(x_1) = \varphi(x_2) = \cdots = \varphi(x_n) = 0.$$

Moreover, from (12)

$$\varphi(x) = 0.$$

Now according to ROLLE's theorem, in each of the n intervals between the $n + 1$ points $x, x_1, x_2, \ldots, x_n$ there is at least one root of the derivative $\varphi'(z)$; $\varphi'(z)$ therefore has at least n distinct roots.

Repeated application of ROLLE's theorem then yields $n - 1$ (distinct!) roots of the second derivative $\varphi''(z)$ in the $n - 1$ intervals between the n roots of $\varphi'(z)$. By continuing this procedure, we arrive at a root of the nth derivative $\varphi^{(n)}(z)$ which lies between the smallest and largest of the numbers $x, x_1, x_2, \ldots, x_n$. Denoting this root by ξ, we then obtain from (13)

$$K = \frac{f^{(n)}(\xi)}{n!}.$$

Formula (12) now leads to the LAGRANGE *interpolation formula with remainder*:

$$f(x) = L(x) + \frac{f^{(n)}(\xi)}{n!}\,\omega(x); \tag{14}$$

it is hereby of importance that $a < \xi < b$.

Formula (14) affords the simple

Theorem. *If $f(x)$ is an entire function defined on $[a, b]$ and if the number of nodes, which are all assumed to lie in $[a, b]$, is increased without bound according to any rule whatever, then*

$$\lim_{n\to\infty} L(x) = f(x)$$

uniformly on $[a, b]$.

Proof. For any $x \in [a, b]$, $|\omega(x)| \leqq (b - a)^n$. If we set

$$M_n = \max |f^{(n)}(x)|,$$

then it follows from (14) that

$$|f(x) - L(x)| \leqq \frac{M_n}{n!}(b - a)^n.$$

In Volume I, Chapter IX, § 1 we proved that

$$\lim_{n\to\infty} \frac{\sqrt[n]{M_n}}{n} = 0.$$

From this it follows that

$$\lim_{n\to\infty}\left[\frac{\sqrt[n]{M_n}}{n}\,e(b - a)\right] = 0,$$

and hence *a fortiori* that

$$\lim_{n\to\infty}\left[\frac{M_n}{n^n}\,e^n(b - a)^n\right] = 0. \tag{15}$$

Since

$$\frac{n^n}{n!} < e^n,$$

it follows from (15) that

$$\lim_{n\to\infty}\left[\frac{M_n}{n!}(b - a)^n\right] = 0,$$

wherewith the theorem is proved.

§3. A Rearrangement of the LAGRANGE Formula—NEWTON's Formula

Suppose that we know the values $f(x_1), \ldots, f(x_n)$ of some function at the nodes $x_1, x_2, \ldots, x_n$, and we now wish to find its values at non-nodal points. If the structural properties of the function are nice enough, then—as we already know—the LAGRANGE interpolation polynomial represents the function to arbitrary accuracy when the number of nodes is sufficiently high. Hence in this case it is both natural and justified to set the unknown value of $f(x)$ equal to the known value of $L(x)$.

For example, suppose that $f(x)$ is the steam pressure in a boiler at temperature x. If we measure this pressure at the temperatures $x_1, x_2, \ldots, x_n$ and form the interpolation polynomial we obtain a formula that makes it possible for us to calculate the pressure at unobserved temperatures as well.[3] However, for this example the form (10) of the interpolation polynomial turns out to be unsuitable. For if we subsequently perform still another measurement of the temperature x_{n+1}, we must then change all summands in the sum (10) and carry out all calculations anew. This situation gave rise to the idea, which goes back to NEWTON, of writing the polynomial $L(x)$ not in the form (10), but rather in the form

$$L(x) = A_0 + A_1(x - x_1) + A_2(x - x_1)(x - x_2) + \cdots + A_{n-1}(x - x_1)\ldots(x - x_{n-1}). \tag{16}$$

If we here substitute $x = x_1, x = x_2, \ldots, x = x_n$ in succession and each time make use of $L(x_i) = y_i$ we thus find all the coefficients $A_0, A_1, \ldots, A_{n-1}$. It is immediately obvious that A_{k-1} depends only on $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_k$ and not on x_i and y_i for $i > k$. Thus introduction of a new node requires the addition of only a single new summand in (16) while all summands already present are retained.

We now develop a formula for calculating A_{k-1}. Since the polynomial

$$L_k(x) = A_0 + A_1(x - x_1) + \cdots + A_{k-1}(x - x_1)\cdots(x - x_{k-1})$$

takes on the values $y_1, y_2, \ldots, y_k$ for the arguments $x = x_1, x_2, \ldots, x_k$, it may be represented in the LAGRANGE form (5):

$$L_k(x) = \sum_{i=1}^{k} \frac{\omega_k(x)}{\omega_k'(x_i)(x - x_i)} y_i,$$

where

$$\omega_k(x) = (x - x_1)(x - x_2)\ldots(x - x_k).$$

Its leading coefficient A_{k-1} is therefore

$$A_{k-1} = \sum_{i=1}^{k} \frac{y_i}{\omega_k'(x_i)}. \tag{17}$$

It remains only to note that

$$\omega_k'(x_i) = (x_i - x_1)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_k). \tag{18}$$

Thus, for example,

$$A_0 = y_1, \quad A_1 = \frac{y_1}{x_1 - x_2} + \frac{y_2}{x_2 - x_1},$$

$$A_2 = \frac{y_1}{(x_1 - x_2)(x_1 - x_3)} + \frac{y_2}{(x_2 - x_1)(x_2 - x_3)} + \frac{y_3}{(x_3 - x_1)(x_3 - x_2)}.$$

[3] Intentionally schematic. In practice direct interpolation is seldom used to obtain an "empirical formula".

We now consider more closely the important case in which the nodes form an arithmetic sequence. For this purpose we first define the *concept of differences*. Let

$$y_0, y_1, y_2, y_3, \ldots \tag{19}$$

be any finite or infinite sequence of numbers. We define [4]

$$\begin{aligned} \Delta y_k &= y_{k+1} - y_k, \\ \Delta^2 y_k &= \Delta y_{k+1} - \Delta y_k, \\ &\cdots\cdots\cdots \\ \Delta^{n+1} y_k &= \Delta^n y_{k+1} - \Delta^n y_k \\ &\cdots\cdots\cdots \end{aligned}$$

One easily sees that

$$\begin{aligned} \Delta^2 y_k &= y_{k+2} - 2y_{k+1} + y_k, \\ \Delta^3 y_k &= y_{k+3} - 3y_{k+2} + 3y_{k+1} - y_k, \end{aligned}$$

and in general

$$\Delta^n y_k = \sum_{r=0}^{n} (-1)^{n-r} C_n^r y_{k+r}, \tag{20}$$

which can easily be verified by complete induction. The quantities Δy_k, $\Delta^2 y_k, \ldots$ are called differences of first, second, etc. order of the sequence (19).

We now return to formula (16) and choose the interpolation nodes

$$x_1 = a, \quad x_2 = a + h, \quad x_3 = a + 2h, \quad \ldots, \quad x_n = a + (n - 1)h,$$

where h is a nonzero number.

In this case

$$x_i - x_r = (i - r)h;$$

hence from (18) it follows that

$$\omega_k'(x_i) = (-1)^{k-i} h^{k-1} (i - 1)!(k - i)!.$$

Substituting this into (17), we find

$$A_{k-1} = \sum_{i=1}^{k} \frac{(-1)^{k-i} y_i}{h^{k-1}(i - 1)!(k - i)!}$$

or

$$A_{k-1} = \frac{1}{h^{k-1}(k - 1)!} \sum_{r=0}^{k-1} (-1)^{k-1-r} C_{k-1}^r y_{r+1}.$$

[4] We actually introduce new variables $\Delta y, \Delta^2 y, \ldots$ starting with the variable y. It would therefore be more natural to adopt the notation $(\Delta y)_k, (\Delta^2 y)_k, \ldots$

Comparing this result with (20), we finally obtain

$$A_{k-1} = \frac{\Delta^{k-1} y_1}{h^{k-1}(k-1)!},$$

and formula (16) thus becomes

$$L(x) = y_1 + \frac{\Delta y_1}{h}\frac{x-a}{1!} + \frac{\Delta^2 y_1}{h^2}\frac{(x-a)(x-a-h)}{2!} + \cdots + \frac{\Delta^{n-1} y_1}{h^{n-1}}\frac{(x-a)(x-a-h)\ldots[x-a-(n-2)h]}{(n-1)!}. \quad (21)$$

This formula is called the NEWTON *interpolation formula.*

If

$$y_k = f[a + (k-1)h],$$

one also uses the notation

$$\Delta^n y_k = \Delta^n f[a + (k-1)h].$$

NEWTON's formula then assumes the form [5]

$$L[f; x] = \sum_{k=0}^{n-1} \frac{\Delta^k f(a)}{h^k}\frac{(x-a)(x-a-h)\ldots[x-a-(k-1)h]}{k!} \quad (22)$$

If, in particular, $P(x)$ is a polynomial from H_{n-1}, then for any pair of numbers a, h the identity

$$P(x) = \sum_{k=0}^{n-1} \frac{\Delta^k P(a)}{h^k}\frac{(x-a)(x-a-h)\ldots[x-a-(k-1)h]}{k!} \quad (23)$$

holds.

Example. Let

$$P(x) = \frac{(n-x)(n-1-x)\ldots(2-x)}{n!}, \quad a = 0,\ h = 1.$$

In this case

$$P(a) = 1, \quad P(a+h) = \frac{1}{n}, \quad P(a+2h) = \cdots = P[a+(n-1)h] = 0,$$

and hence

$$\Delta^k P(a) = \sum_{r=0}^{k} (-1)^{k-r} C_k^r P(a+rh) = (-1)^k \frac{n-k}{n}.$$

[5] Here $\Delta^0 y_k = y_k$, $\Delta^0 f(a) = f(a)$.

Therefore

$$\frac{(n-x)(n-1-x)\ldots(2-x)}{n!} = \sum_{k=0}^{n-1} (-1)^k \frac{n-k}{n}\frac{x(x-1)\ldots(x-k+1)}{k!}.$$

Substituting $x = n + m$, we obtain the useful identity

$$\sum_{k=0}^{n-1} (-1)^k \frac{n-k}{n} C_{n+m}^k = (-1)^{n-1}\frac{(n+m-2)!}{(m-1)!n!}. \quad (24)$$

We shall subsequently make use of this identity in a somewhat different form; this is obtained by replacing the index k by $n-k$ and interchanging the roles of n and m:

$$\sum_{i=0}^{m} (-1)^{m-i}\frac{i}{n} C_{n+m}^{n+i} = (-1)^{m-1}\frac{(m+n-2)!}{(m-1)!n!}. \quad (25)$$

§4. Interpolation with Multiple Nodes

In the preceding sections we have constructed the interpolation polynomial from its values at the nodal points. We now consider the following more general problem. Suppose that we are given the nodes (1) and the numbers

$$\begin{array}{llll} y_1, & y_1', & \ldots, & y_1^{(\alpha_1-1)}, \\ y_2, & y_2', & \ldots, & y_2^{(\alpha_2-1)}, \\ \cdots & & & \\ \cdots & & & \\ y_n, & y_n', & \ldots, & y_n^{(\alpha_n-1)}. \end{array}$$

It is then required to construct a polynomial $H(x)$ of lowest possible degree which satisfies the conditions

$$H^{(r)}(x_i) = y_i^{(r)} \qquad (i = 1, 2, \ldots, n;\ r = 0, 1, \ldots, \alpha_i - 1). \quad (26)$$

This sort of interpolation was first studied by HERMITE [2].

It is easily seen that the problem has exactly one solution. Indeed, if we set

$$P_i(x) = A_0^{(i)} + A_1^{(i)}(x - x_i) + \cdots + A_{\alpha_i-1}^{(i)}(x - x_i)^{\alpha_i-1},$$

then

$$H(x) = P_1(x) + (x-x_1)^{\alpha_1}P_2(x) + \cdots + (x-x_1)^{\alpha_1}(x-x_2)^{\alpha_2}\ldots(x-x_{n-1})^{\alpha_{n-1}}P_n(x). \quad (27)$$

For the point x_1 is an α_1-fold root of $H(x) - P_1(x)$; differentiating (27) $(\alpha_1 - 1)$ times in succession and substituting $x = x_1$ into the expressions so obtained and into (27), we obtain all the coefficients of the polynomial $P_1(x)$. In the same way we then make use of the equation

$$\frac{H(x) - P_1(x)}{(x - x_1)^{\alpha_1}} = P_2(x) + \cdots + (x - x_2)^{\alpha_2} \ldots (x - x_{n-1})^{\alpha_{n-1}} P_n(x)$$

in order to determine the coefficients of the polynomial $P_2(x)$, and etc. Finally, all the coefficients of $H(x)$ are determined. The degree of $H(x)$ is clearly not greater than $m - 1$ where $m = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. On the other hand, there is no other polynomial $M(x)$ from H_{m-1} which satisfies all the conditions (26), since otherwise the difference $H(x) - M(x)$ would have a total of m roots (taking into account their multiplicities).

It is possible to find a formula which expresses the coefficients of the polynomial $H(x)$ in terms of the values given in the problem; however, we shall not here undertake these general considerations, but rather restrict ourselves to three special cases:

I. If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$ the problem leads to construction of the LAGRANGE interpolation polynomial.

II. If $n = 1$ there is only one node, and the solution to the problem is given by the TAYLOR polynomial

$$H(x) = y_1 + \frac{y_1'}{1!}(x - x_1) + \cdots + \frac{y_1^{(\alpha_1-1)}}{(\alpha_1 - 1)!}(x - x_1)^{\alpha_1 - 1}.$$

III. If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 2$ the solution reads

$$H(x) = \sum_{k=1}^{n} y_k \left[1 - \frac{\omega''(x_k)}{\omega'(x_k)}(x - x_k)\right] l_k^2(x) + \sum_{k=1}^{n} y_k'(x - x_k) l_k^2(x); \quad (28)$$

where (as previously)

$$\omega(x) = (x - x_1)(x - x_2) \ldots (x - x_n), \quad l_k(x) = \frac{\omega(x)}{\omega'(x_k)(x - x_k)}.$$

To prove formula (28) we first note that the degree of the polynomial $H(x)$ is not greater than $(2n - 1)$. The equations

$$H(x_i) = y_i, \quad H'(x_i) = y_i' \quad (i = 1, 2, \ldots, n) \quad (29$$

are also easily verified. For

$$l_k'(x) = \frac{\omega'(x)(x - x_k) - \omega(x)}{\omega'(x_k)(x - x_k)^2},$$

and hence according to L'HOPITAL's rule

$$l_k'(x_k) = \lim_{x \to x_k} l_k'(x) = \lim_{x \to x_k} \frac{\omega''(x)(x - x_k) + \omega'(x) - \omega'(x)}{2\omega'(x_k)(x - x_k)} = \frac{\omega''(x_k)}{2\omega'(x_k)}.$$

The polynomial $q_k(x) = l_k^2(x)$ therefore satisfies the conditions

$$q_k(x_i) = \begin{cases} 0 & (i \neq k), \\ 1 & (i = k); \end{cases} \qquad q_k'(x_i) = \begin{cases} 0 & (i \neq k), \\ \dfrac{\omega''(x_k)}{\omega'(x_k)} & (i = k). \end{cases}$$

From this it follows that for the polynomials

$$A(x) = \sum_{k=1}^{n} y_k \left[1 - (x - x_k)\frac{\omega''(x_k)}{\omega'(x_k)}\right] q_k(x),$$

$$B(x) = \sum_{k=1}^{n} y_k'(x - x_k) q_k(x)$$

the equations

$$A(x_i) = y_i, \qquad A'(x_i) = 0,$$
$$B(x_i) = 0, \qquad B'(x_i) = y_i'$$

hold; this is however equivalent to (29).

We further write formula (28) in the form

$$H(x) = \sum_{k=1}^{n} y_k A_k(x) + \sum_{k=1}^{n} y_k' B_k(x) \quad (30)$$

with

$$\left.\begin{aligned} A_k(x) &= \left[1 - \frac{\omega''(x_k)}{\omega'(x_k)}(x - x_k)\right] l_k^2(x), \\ B_k(x) &= (x - x_k) l_k^2(x). \end{aligned}\right\} \quad (31)$$

Let us now return to the general case. Suppose $f(x)$ is a function defined on $[a, b]$ which has there a finite derivative of order $m = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ where $\alpha_k \geqq 1$.

Now let $x_1, x_2, \ldots x_n$ be nodes lying in $[a, b]$; we put

$$y_i^{(r)} = f^{(r)}(x_i) \qquad (i = 1, 2, \ldots, n; \quad r = 0, 1, \ldots, \alpha_i - 1).$$

We can now construct the HERMITE interpolation polynomial in accordance with conditions (26) and then study the difference

$$f(x) - H(x),$$

where x is a fixed point of $[a, b]$ which is not a node x_i.

We now define

$$\Omega(z) = (z - x_1)^{a_1}(z - x_2)^{a_2} \dots (z - x_n)^{a_n}$$

and introduce the function

$$\varphi(z) = f(z) - H(z) - K\Omega(z),$$

where

$$K = \frac{f(x) - H(x)}{\Omega(x)}. \qquad (32)$$

Then

$$\varphi^{(r)}(x_i) = 0 \qquad (i = 1, 2, \dots, n; \quad r = 0, 1, \dots, a_i - 1),$$

for x_i is an α_i-fold root of $\Omega(z)$. Moreover, (32) implies that

$$\varphi(x) = 0.$$

The function $\varphi(z)$ therefore has at least $m + 1$ roots in $[a, b]$ (taking into account their multiplicities). From this on the basis of ROLLE's theorem we find at least m roots for $\varphi'(z)$, at least $m - 1$ roots for $\varphi''(z)$, and etc. In particular, $\varphi^{(m)}(z)$ has at least one root ξ. But now

$$\varphi^{(m)}(z) = f^{(m)}(z) - Km!,$$

whence

$$K = \frac{f^{(m)}(\xi)}{m!},$$

and it follows that

$$f(x) = H(x) + \frac{f^{(m)}(\xi)}{m!}\,\Omega(x) \qquad (a < \xi < b).$$

This is the HERMITE *interpolation formula with remainder*. It follows from this formula, in a manner analogous to what was done previously, that the HERMITE polynomials approximate an entire function uniformly when the number of nodes (which may be distributed in any fashion) increases without bound.

§5. Trigonometric Interpolation

Suppose the $2n + 1$ points

$$x_0, x_1, x_2, \dots, x_{2n} \qquad (33)$$

are given in the half-open interval $[0, 2\pi)$. It is then easy to construct a

trigonometric polynomial $T(x)$ of lowest possible order which assumes preassigned values $y_1, y_2, \dots, y_{2n}$ at the nodes (33).

Since

$$\sin\frac{x-a}{2}\sin\frac{x-b}{2} = \frac{1}{2}\left[\cos\frac{b-a}{2} - \cos\left(x - \frac{a+b}{2}\right)\right]$$

is a trigonometric polynomial of first order,

$$t_k(x) = \frac{\sin\frac{x-x_0}{2}\dots\sin\frac{x-x_{k-1}}{2}\sin\frac{x-x_{k+1}}{2}\dots\sin\frac{x-x_{2n}}{2}}{\sin\frac{x_k-x_0}{2}\dots\sin\frac{x_k-x_{k-1}}{2}\sin\frac{x_k-x_{k+1}}{2}\dots\sin\frac{x_k-x_{2n}}{2}} \qquad (34)$$

is a trignometric polynomial of nth order (it is important that the number of factors of the numerator be even).

Clearly

$$t_k(x_i) = \begin{cases} 0 & \text{for } i \neq k \\ 1 & \text{for } i = k. \end{cases}$$

Hence the polynomial

$$T(x) = \sum_{k=0}^{2n} y_k t_k(x) \qquad (35)$$

satisfies the conditions

$$T(x_i) = y_i \qquad (i = 0, 1, \dots, 2n). \qquad (36)$$

The order of $T(x)$ is at most n. There can be no other polynomials $M(x)$ from H_n^T which satisfy the given conditions, for otherwise the difference $T(x) - M(x)$ would be a polynomial of H_n^T, not identically zero, which would have the $2n + 1$ roots (33); this however is a contradiction.

Suppose now that $n + 1$ nodes

$$x_0, x_1, \dots, x_n \qquad (37)$$

are given in the interval $[0, \pi]$; if we put

$$c_k(x) = \frac{\left\{\begin{matrix}(\cos x - \cos x_0)\dots(\cos x - \cos x_{k-1})(\cos x - \cos x_{k+1})\\ \dots(\cos x - \cos x_n)\end{matrix}\right\}}{\left\{\begin{matrix}(\cos x_k - \cos x_0)\dots(\cos x_k - \cos x_{k-1})(\cos x_k - \cos x_{k+1})\\ \dots(\cos x_k - \cos x_n)\end{matrix}\right\}}$$

$$C(x) = \sum_{k=0}^{n} y_k c_k(x),$$

The interpolation polynomial for the nodes (40) therefore has the form

$$T(x) = \frac{1}{2n+1} \sum_{k=0}^{2n} y_k \frac{\sin \frac{2n+1}{2}(x - x_k)}{\sin \frac{x - x_k}{2}}, \tag{42}$$

which is reminiscent of the DIRICHLET integral, and one will note that the behavior of the polynomials $T(x)$ is closely analogous to that of partial FOURIER sums.

We now seek to determine the coefficients A, a_m, b_m in the polynomial (42) by writing it in "canonical form"

$$T(x) = A + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx). \tag{43}$$

To this end, we substitute the expressions for $t_k(x)$ obtained from (41) into (42):

$$T(x) = \frac{1}{2n+1} \sum_{k=0}^{2n} y_k \left[1 + 2 \sum_{m=1}^{n} \cos m(x - x_k) \right].$$

From this it follows that

$$T(x) = \frac{1}{2n+1} \left(\sum_{k=0}^{2n} y_k \right) + \frac{2}{2n+1} \sum_{m=1}^{n} \left[\left(\sum_{k=0}^{2n} y_k \cos mx_k \right) \cos mx + \left(\sum_{k=0}^{2n} y_k \sin mx_k \right) \sin mx \right] \tag{44}$$

and hence

$$A = \frac{1}{2n+1} \sum_{k=0}^{2n} y_k,$$

$$a_m = \frac{2}{2n+1} \sum_{k=0}^{2n} y_k \cos mx_k, \qquad b_m = \frac{2}{2n+1} \sum_{k=0}^{2n} y_k \sin mx_k.$$

Now if $y_0, y_1, \ldots, y_{2n}$ are the values of some function $f(x)$ at the nodes (40), then the values of the coefficients A, a_m, b_m obtained are nothing but RIEMANN sums for the FOURIER coefficients

$$\frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx, \qquad \frac{1}{\pi} \int_0^{2\pi} f(x) \cos mx\, dx, \qquad \frac{1}{\pi} \int_0^{2\pi} f(x) \sin mx\, dx. \tag{45}$$

As n becomes large, the coefficients A, a_m, b_m approach the integrals (45), and the polynomial $T(x)$ approaches the FOURIER sum $S_n(x)$ of the function $f(x)$. This is, of course, intended to be only a heuristic consideration and not a precisely formulated theorem.

CHAPTER II

THEOREMS OF NEGATIVE CHARACTER

§1. The Theorems of S. N. BERNSTEIN and G. FABER

Let us consider the following problem: In the interval $[a, b]$ we choose nodes which form an infinite triangular matrix:

$$\left.\begin{array}{l} x_1^{(1)}, \\ x_1^{(2)}, x_2^{(2)}, \\ x_1^{(3)}, x_2^{(3)}, x_3^{(3)}, \\ \cdots\cdots\cdots\cdots\cdots \\ x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \ldots, x_n^{(n)} \\ \cdots\cdots\cdots\cdots\cdots \end{array}\right\} \tag{46}$$

Given a function $f(x)$ defined on $[a, b]$, we then construct a sequence $\{L_n(x)\}$ of LAGRANGE interpolation polynomials, where in so doing we use as interpolation nodes in the construction of $L_n(x)$ the n elements of the nth row of the matrix (46), such that

$$L_n(x_k^{(n)}) = f(x_k^{(n)}) \qquad (k = 1, 2, \ldots, n).$$

One then asks the question: Will $L_n(x)$ in this case converge to $f(x)$ at all points of the interval $[a, b]$?

As we already know, the sequence does converge (and even uniformly) for entire functions $f(x)$. It is desirable to avoid such a strong condition. It will now be shown that to every matrix (46) there corresponds a class of functions for which the interpolation process obtained from the matrix converges uniformly.[1] However, this class is always substantially more restricted than the class $C([a, b])$ of all continuous functions on $[a, b]$. In other words, there is no universal matrix (46) which is applicable for all continuous functions. The proof of this fact is the substance of FABER's theorem to which this section is primarily devoted.

[1] This class is never empty, since it contains all entire functions (and—trivially—all polynomials).

In studying questions of convergence of the polynomials $L_n(x)$ to a function $f(x)$, the quantity

$$\lambda_n(x) = \sum_{k=1}^{n} |l_k^{(n)}(x)| \tag{47}$$

plays a major role; here $l_k^{(n)}(x)$ are the basis polynomials of the nth row of the matrix (46), i.e.

$$l_k^{(n)}(x) = \frac{\omega_n(x)}{\omega_n'(x_k^{(n)})(x - x_k^{(n)})} \qquad \left(\omega_n(x) = \prod_{k=1}^{n} (x - x_k^{(n)})\right).$$

The function $\lambda_n(x)$ is the analogue of the Lebesque function which we have studied in the theory of orthogonal polynomials. Putting

$$\lambda_n = \max \lambda_n(x) \qquad (a \leqq x \leqq b), \tag{48}$$

we formulate

Theorem 1 (G. Faber-S. N. Bernstein). *For every matrix* (46) *the inequality*

$$\lambda_n > \frac{\ln n}{8\sqrt{\pi}} \tag{49}$$

holds.

Proof. The proof [2] of this important theorem hinges on two lemmas.

Lemma 1. *Given n arbitrary points* $\theta_1, \theta_2, \ldots, \theta_n$ $(0 \leqq \theta_k \leqq \pi)$, *there exists an even trigonometric polynomial* $T(\theta)$ *of order at most* $n - 1$ *such that*

$$|T(\theta_i)| \leqq 8\sqrt{\pi} \qquad (i = 1, 2, \ldots, n) \tag{50}$$

and which satisfies the inequality

$$T(\alpha) > \ln n \tag{51}$$

at at least one point $\alpha \in [0, \pi]$.

Proof. Let $c_k(\theta)$ be those even trigonometric polynomials of order at most $n - 1$ for which

$$c_k(\theta_i) = \begin{cases} 0 & \text{for} \quad i \neq k, \\ 1 & \text{for} \quad i = k. \end{cases}$$

Further, let

[2] G. Faber [1], S. N. Bernstein [6]. The proof given in the text is due to Fejér [3].

$$A(\theta) = \frac{\cos\theta}{n-1} + \frac{\cos 2\theta}{n-2} + \cdots + \frac{\cos(n-1)\theta}{1},$$

$$B(\theta) = \frac{\cos(n+1)\theta}{1} + \frac{\cos(n+2)\theta}{2} + \cdots + \frac{\cos(2n-1)\theta}{n-1}$$

As we already know [3] the following inequality holds for all θ:

$$|A(\theta) - B(\theta)| \leqq 4\sqrt{\pi}. \tag{52}$$

Finally, let us introduce the polynomial

$$U(\theta) = A(2\theta) - \sum_{k=1}^{n} [B(\theta_k + \theta) + B(\theta_k - \theta)]c_k(\theta),$$

which is an even trigonometric polynomial. It is easily seen that

$$\int_0^{\pi} U(\theta)\, d\theta = \frac{1}{2}\int_{-\pi}^{\pi} U(\theta)\, d\theta = 0. \tag{53}$$

For $A(2\theta)$ is a trigonometric polynomial without constant term which is therefore orthogonal to 1 on the interval $[-\pi, \pi]$. Also, $B(\theta_k + \theta) + B(\theta_k - \theta)$ is a linear combination of terms of the form $\cos m\theta$ with $m > n$; this linear combination is therefore orthogonal to the polynomials $c_k(\theta)$ whose order is less than n.

Hence, there exists a point α in the interval $[0, \pi]$ at which

$$U(\alpha) = 0.$$

With these remarks in mind, we define

$$T(\theta) = [A(\theta + \alpha) + A(\theta - \alpha)] - \sum_{k=1}^{n} [B(\theta_k + \alpha) + B(\theta_k - \alpha)]c_k(\theta).$$

This is also an even trigonometric polynomial from H_{n-1}^T,[4] and in particular

$$T(\theta_i) = [A(\theta_i + \alpha) - B(\theta_i + \alpha)] + [A(\theta_i - \alpha) - B(\theta_i - \alpha)];$$

therefore, on the basis of (52) it satisfies (50). On the other hand,

$$T(\alpha) = A(0) + U(\alpha) = A(0)$$

or

$$T(\alpha) = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} > \int_1^n \frac{dx}{x} = \ln n,$$

and hence (51) is also satisfied.

[3] Volume I, inequality (186).
[4] For $B(\theta_k + \alpha)$ and $B(\theta_k - \alpha)$ are constants.

Lemma 2. *Given n arbitrary nodes $x_1, x_2, \ldots, x_n$ in $[a, b]$, there exists a polynomial $P(x) \in H_{n-1}$ such that*

$$|P(x_i)| \leqq 8\sqrt{\pi} \qquad (i = 1, 2, \ldots, n) \tag{54}$$

and which satisfies the inequality

$$P(c) > \ln n \tag{55}$$

at at least one point $c \in [a, b]$.

Proof. This lemma is easily obtained from the previous one. The substitution

$$\theta = \operatorname{arc\,cos} \frac{2x - (a + b)}{b - a}$$

maps the interval $[a, b]$ onto the interval $[0, \pi]$ and takes the point x_k into the point θ_k. If now $T(x)$ is the trigonometric polynomial whose existence was established in Lemma 1, then the polynomia

$$P(x) = T\left[\operatorname{arc\,cos} \frac{2x - (a+b)}{b - a}\right]$$

satisfies inequalities (54) and (55), where

$$c = \frac{b - a}{2} \cos \alpha + \frac{a + b}{2}.$$

Returning now to the Faber-Bernstein theorem, we note that the polynomial of Lemma 2 can be written in the form

$$P(x) = \sum_{k=1}^{n} P(x_k) l_k(x).$$

From this it follows that

$$|P(x)| \leqq 8\sqrt{\pi} \sum_{k=1}^{n} |l_k(x)|,$$

and therefore from (55) that

$$\sum_{k=1}^{n} |l_k(c)| > \frac{\ln n}{8\sqrt{\pi}}.$$

This holds for arbitrary interpolation nodes in $[a, b]$ and hence in particular for the nodes occurring in the nth row of our matrix. The theorem is herewith proved.

Theorem 2 (G. Faber). *Given a matrix (46) the points of which all lie in $[a, b]$, there exists a function $f(x) \in C([a, b])$ such that the interpolation polynomial formed from the rows of the matrix does not converge uniformly to $f(x)$.*

Proof. The proof is by contradiction. Suppose therefore that there is a matrix (46) which for every function $f(x) \in C([a, b])$ ensures uniform convergence of the interpolation polynomial $L_n[f, x] = L_n(x)$ to $f(x)$. As above, we put

$$\lambda_n(x) = \sum_{k=1}^{n} |l_k^{(n)}(x)|, \qquad \lambda_n = \max \lambda_n(x), \tag{56}$$

and in particular for $z_n \in [a, b]$ let

$$\lambda_n(z_n) = \lambda_n. \tag{57}$$

We now construct a continuous function $\varphi_n(x)$ for every natural number n by putting first of all

$$\varphi_n(x_k^{(n)}) = \operatorname{sign} l_k^{(n)}(z_n) \qquad (k = 1, 2, \ldots, n) \tag{58}$$

and requiring secondly that this function $\varphi_n(x)$ be linear between the nodes $x_k^{(n)}$. In order to define it in the entire interval $[a, b]$, we must still attend to its values in the intervals[5] $[a, x_1^{(n)}]$ and $[x_n^{(n)}, b]$: let the function be constant on both these intervals.

It is clear that everywhere in $[a, b]$

$$|\varphi_n(x)| \leqq 1. \tag{59}$$

Also,

$$L_n[\varphi_n; z_n] = \sum_{k=1}^{n} \varphi_n(x_k^{(n)}) l_k^{(n)}(z_n) = \sum_{k=1}^{n} |l_k^{(n)}(z_n)| = \lambda_n(z_n) = \lambda_n. \tag{60}$$

We now construct an increasing sequence of natural numbers $n_1 < n_2 < n_3 < \cdots$, where in so doing we choose n_1 such that

$$\lambda_{n_1} > 2 \cdot 2 \cdot 3;$$

this is possible, since according to Theorem 1 the numbers λ_n increase without bound. Since the function $\frac{\varphi_{n_1}(x)}{3}$ is continuous, according to our assumption its interpolation process converges uniformly. There therefore exists an index n' such that for all $n > n'$

$$\left| L_n\left[\frac{\varphi_{n_1}}{3}; x\right]\right| < 1.$$

[5] We number the nodes in the natural order

$$a \leqq x_1^{(n)} < x_2^{(n)} < \cdots < x_n^{(n)} \leqq b.$$

We now choose an index n_2 such that $n_2 > n'$, $n_2 > n_1$, and moreover

$$\lambda_{n_2} > 2\cdot 3\cdot 3^2.$$

The function

$$\frac{\varphi_{n_1}(x)}{3} + \frac{\varphi_{n_2}(x)}{3^2}$$

is then continuous and is less in absolute value than $\frac{1}{3} + \frac{1}{3^2} < 1$. We can therefore find an index n'' such that for $n > n''$

$$\left| L_n\left[\frac{\varphi_{n_1}}{3} + \frac{\varphi_{n_2}}{3^2}\,;\, x\right]\right| < 1.$$

We now choose $n_3 > n''$, $n_3 > n_2$ such that in addition

$$\lambda_{n_3} > 2\cdot 4\cdot 3^3.$$

Continuing this process, we obtain the desired sequence $\{n_m\}$, where for every index m the inequalities

$$\left| L_{n_m}\left[\frac{\varphi_{n_1}}{3} + \frac{\varphi_{n_2}}{3^2} + \cdots + \frac{\varphi_{n_{m-1}}}{3^{m-1}}\,;\, x\right]\right| < 1, \tag{61}$$

$$\lambda_{n_m} > 2(m+1)\,3^m \tag{62}$$

hold. Making use of the numbers n_m, we now define the function

$$f(x) = \sum_{k=1}^{\infty} \frac{\varphi_{n_k}(x)}{3^k},$$

the continuity of which is obvious. If we define

$$A(x) = \sum_{k=1}^{m-1} \frac{\varphi_{n_k}(x)}{3^k}, \qquad B(x) = \sum_{k=m+1}^{\infty} \frac{\varphi_{n_k}(x)}{3^k},$$

then

$$f(x) = A(x) + \frac{\varphi_{n_m}(x)}{3^m} + B(x);$$

whence it follows that

$$L_{n_m}[f; x] = L_{n_m}[A; x] + \frac{1}{3^m} L_{n_m}[\varphi_{n_m}; x] + L_{n_m}[B; x].$$

According to (61) we have

$$|L_{n_m}[A; x]| < 1. \tag{63}$$

Moreover,

$$|L_{n_m}[B; x]| = \left| \sum_{k=1}^{n_m} B(x_k^{(n_m)}) l_k^{(n_m)}(x) \right| \leqq \max |B(x)| \sum_{k=1}^{n_m} |l_k^{(n_m)}(x)|,$$

and hence

$$|L_{n_m}[B; x]| \leqq \lambda_{n_m} \max |B(x)|.$$

But now

$$|B(x)| \leqq \frac{1}{3^{m+1}} + \frac{1}{3^{m+2}} + \cdots = \frac{1}{2\cdot 3^m}, \tag{64}$$

and hence

$$|L_{n_m}[B; x]| \leqq \frac{\lambda_{n_m}}{2\cdot 3^m}. \tag{65}$$

Moreover,

$$L_{n_m}[f; z_{n_m}] \geqq \frac{1}{3^m} L_{n_m}[\varphi_{n_m}; z_{n_m}] - |L_{n_m}[A; z_{n_m}]| - |L_{n_m}[B; z_{n_m}]|;$$

hence, according to (60), (63), and (65) we obtain

$$L_{n_m}[f; z_{n_m}] > \frac{\lambda_{n_m}}{3^m} - 1 - \frac{\lambda_{n_m}}{2\cdot 3^m} = \frac{\lambda_{n_m}}{2\cdot 3^m} - 1$$

and from this according to (62)

$$L_{n_m}[f; z_{n_m}] > m. \tag{66}$$

Hence

$$\lim_{m\to\infty} L_{n_m}[f; z_{n_m}] = +\infty, \tag{67}$$

and this precludes uniform convergence of $L_n[f;x]$ to $f(x)$ contrary to our assumption.[6]

The FABER theorem shows that for every matrix (46) there exists a continuous function $f(x)$, constructed with the help of the matrix, the interpolation process for which does not converge uniformly. The question if

[6] This method of proof is used very often. I suggest that it be called the method of the "sliding hump."

there is perhaps such a function applicable to all such matrices is then of interest. This question is answered negatively by

Theorem 3 (J. MARCINKIEWICZ [1]). *For every continuous function $f(x)$ there exists a matrix (46) such that the interpolation polynomial obtained from it converges uniformly to $f(x)$.*

Proof. If of all the polynomials of H_{n-1}, $P_{n-1}(x)$ is the one of smallest deviation from $f(x)$, then there exists an $(n+1)$-termed TCHEBYSHEFF alternant consisting of the points $y_1 < y_2 < \cdots < y_{n+1}$ at which the difference $P_{n-1}(x) - f(x)$ has alternating sign. Now in each interval (y_k, y_{k+1}) there is a root $x_k^{(n)}$ of the difference in question. We take these roots as the nodes of the nth row of our matrix (46). The polynomial $P_{n-1}(x)$ is then at the same time the interpolation polynomial for $f(x)$ corresponding to the nodes $x_k^{(n)}$, and we have only to show that the polynomial $P_{n-1}(x)$ converges uniformly to $f(x)$.

§2. The BERNSTEIN Example

The FABER theorem states that the interpolation polynomial $L_n(x)$ does not always converge uniformly to $f(x)$. This does not exclude the possibility that the sequence of polynomials $L_n(x)$ converge to $f(x)$ at many (or even all) points. The following example shows that the interpolation process may diverge everywhere with the exception of single points.

Theorem (S. N. BERNSTEIN). *The interpolation polynomial $L_n(x)$ for the function $|x|$ with uniformly distributed nodes in the interval $[-1, +1]$ (such that $x_1 = -1$, $x_n = +1$) converges to $|x|$ at no point of the interval $[-1, +1]$ with the exception*[7] *of the points -1, 0, and $+1$.*

Proof. We prove the theorem for a point x such that $-1 < x < 0$; the case where $0 < x < 1$ is entirely analogous.

We introduce the function

$$\varphi(x) = \begin{cases} 0 & \text{for} \quad -1 \leqq x \leqq 0. \\ x & \text{for} \quad 0 \leqq x \leqq 1. \end{cases}$$

Since $x = 2\varphi(x) - x$, it is sufficient to prove divergence of the interpolation process for the function $\varphi(x)$.

[7] For the points ± 1 this is clear, since they are nodal points for every value of n.

We choose the $2n+1$ nodes

$$x_k = -1 + \frac{k-1}{n} \qquad (k = 1, 2, \ldots, 2n+1) \quad (68)$$

and denote the interpolation polynomial for the function $\varphi(x)$ which corresponds to them by $L_{2n+1}(x)$.

From NEWTON's formula (22) we have

$$L_{2n+1}(x) = \sum_{k=0}^{2n} n^k \frac{\Delta^k \varphi(-1)}{k!}(x - x_1)(x - x_2)\ldots(x - x_k) \qquad (69)$$

and from (20)

$$\Delta^k \varphi(-1) = \sum_{r=0}^{k} (-1)^{k-r} C_k^r \varphi\left(-1 + \frac{r}{n}\right).$$

Now for $r \leqq n$, $\varphi\left(-1 + \frac{r}{n}\right) = 0$, and hence

$$\Delta^k \varphi(-1) = 0 \qquad (k = 0, 1, 2, \ldots, n).$$

If however $r = n + i$ $(i = 1, 2, \ldots, n)$, then $\varphi\left(-1 + \frac{r}{n}\right) = \frac{i}{n}$, and hence for $k = n + m$ $(m = 1, 2, \ldots, n)$

$$\Delta^{n+m} \varphi(-1) = \sum_{i=1}^{m} (-1)^{m-i} C_{n+m}^{n+i} \frac{i}{n}$$

or from (25)

$$\Delta^{n+m} \varphi(-1) = (-1)^{m-1} \frac{(m+n-2)!}{(m-1)!\,n!}.$$

Equation (69) now becomes

$$L_{2n+1}(x) = \sum_{m=1}^{n} (-1)^{m-1} \frac{(m+n-2)!}{(m-1)!\,n!} \frac{n^{n+m}}{(n+m)!}(x+1) \times \left(x + \frac{n-1}{n}\right)\ldots\left(x + \frac{1}{n}\right) x \left(x - \frac{1}{n}\right)\ldots\left(x - \frac{m-1}{n}\right). \quad (70)$$

Here all summands have the same sign (as we shall soon show). Let

$$-\frac{i+1}{n} < x \leqq -\frac{i}{n}. \qquad (71)$$

The product

$$(x+1)\left(x + \frac{n-1}{n}\right)\ldots\left(x + \frac{1}{n}\right) x \left(x - \frac{1}{n}\right)\ldots\left(x - \frac{m-1}{n}\right) \quad (72)$$

has $n + m$ factors. Among thse the $m + i$ factors

$$\left(x + \frac{i}{n}\right),\ \left(x + \frac{i-1}{n}\right),\ \ldots,\ \left(x + \frac{1}{n}\right),\ x,\ \left(x - \frac{1}{n}\right),\ \ldots,\ \left(x - \frac{m-1}{n}\right)$$

are negative, and the rest are positive. The sign of the product is therefore $(-1)^{m+i}$. Therefore, all the summands in (70) have the same sign; the absolute value of this sum is therefore greater than the absolute value of the last summand, i.e.

$$|L_{2n+1}(x)| \geqq \frac{\left|(x+1)\left(x+1-\frac{1}{n}\right)\cdots\left(x - \frac{n-1}{n}\right)\right|}{2(2n-1)(n!)^2} n^{2n}.$$

From (71)

$$x = -\frac{i}{n} - \frac{\theta_n}{n} \qquad (0 \leqq \theta_n < 1);$$

so that we obtain

$$|L_{2n+1}(x)| \geqq \frac{n^{2n}}{2(2n-1)(n!)^2}\left(\frac{n-i}{n} - \frac{\theta_n}{n}\right)\left(\frac{n-i-1}{n} - \frac{\theta_n}{n}\right)$$
$$\cdots\left(\frac{1}{n} - \frac{\theta_n}{n}\right)\frac{\theta_n}{n}\left(\frac{1}{n} + \frac{\theta_n}{n}\right)\cdots\left(\frac{n+i-1}{n} + \frac{\theta_n}{n}\right).$$

The righthand side of this inequality becomes no greater if we replace θ_n by 1 in the first $n - i - 1$ factors and by zero in the last $n + i - 1$ factors. Hence

$$|L_{2n+1}(x)| \geqq \frac{(n-i-1)!(n+i-1)!}{2(2n-1)(n!)^2}\theta_n(1-\theta_n).$$

Let us examine the factor

$$\sigma_n = \frac{(n-i-1)!(n+i-1)!}{2(2n-1)(n!)^2}$$

more closely.

It is clear that

$$\sigma_n = \frac{1}{4n-2}\,\frac{1}{n-1}\,\frac{1}{n}\left(1 + \frac{i+1}{n-i}\right)\left(1 + \frac{i+1}{n-i+1}\right)\cdots\left(1 + \frac{i+1}{n-2}\right)$$

Of the $i - 1$ terms in parentheses the last is the smallest. Hence

$$\sigma_n > \frac{1}{4n^3}\left(1 + \frac{i+1}{n-2}\right)^{i-1}.$$

But from (71)

$$\frac{i+1}{n-2} > -x, \quad i - 1 > -nx - 2,$$

so that

$$\sigma_n > \frac{1}{4n^3}(1-x)^{-nx-2}, \tag{73}$$

whence it follows that

$$\lim_{n\to\infty} \sigma_n = +\infty. \tag{74}$$

Until now n was an arbitrary natural number. We now make a special choice for it. To this end we fix a number q which satisfies the condition

$$0 < q < \frac{1+x}{2}.$$

Then for every natural number i the length of the interval $\left(\frac{i+q}{-x}, \frac{i+1-q}{-x}\right)$ is greater than one so that it contains at least one natural number n:

$$\frac{i+q}{-x} < n < \frac{i+1-q}{-x}.$$

For this number n (which by appropriate choice of i can be made arbitrarily large)

$$\frac{i}{n} + \frac{q}{n} < -x < \frac{i+1}{n} - \frac{q}{n},$$

whence

$$\theta_n > q, \quad 1 - \theta_n > q$$

and finally

$$|L_{2n+1}(x)| > q^2\sigma_n.$$

This in conjunction with (74) completes the proof of the theorem.

Addendum. *The interpolation process of the theorem converges to the function* $|x|$ *at the point* $x = 0$.

Proof. If the number of nodes is odd, then the point $x = 0$ is itself a node. If the number of nodes is even and equal to $2n$:

$$x_k = -1 + \frac{2(k-1)}{2n-1} \qquad (k = 1, 2, \ldots, 2n),$$

and then the LAGRANGE interpolation polynomial corresponding to these nodes is

$$L_{2n}(x) = \sum_{k=1}^{2n} |x_k| \frac{(x - x_1)\ldots(x - x_{k-1})(x - x_{k+1})\ldots(x - x_{2n})}{(x_k - x_1)\ldots(x_k - x_{k-1})(x_k - x_{k+1})\ldots(x_k - x_{2n})};$$

hence

$$|L_{2n}(0)| \leqq \sum_{k=1}^{2n} \frac{|x_1 x_2 x_3 \ldots x_{2n}|}{|x_k - x_1|\ldots|x_k - x_{k-1}|\,|x_k - x_{k+1}|\ldots|x_k - x_{2n}|}.$$

Noting that

$$x_k - x_i = \frac{2(k - i)}{2n - 1},$$

we find

$$|L_{2n}(0)| \leqq |x_1 x_2 \ldots x_{2n}| \sum_{k=1}^{2n} \frac{(2n - 1)^{2n-1}}{2^{2n-1}(k - 1)!(2n - k)!}.$$

On the other hand

$$|x_1 x_2 \ldots x_{2n}| = \frac{[(2n - 1)!!]^2}{(2n - 1)^{2n}},$$

and hence

$$|L_{2n}(0)| \leqq \frac{[(2n - 1)!!]^2}{2^{2n-1}(2n - 1)} \sum_{k=1}^{2n} \frac{1}{(k - 1)!(2n - k)!}$$

or equivalently

$$|L_{2n}(0)| \leqq \frac{[(2n - 1)!!]^2}{2^{2n-1}(2n)!(2n - 1)} \sum_{k=1}^{2n} kC_{2n}^k.$$

From STIRLING's formula [8]

$$n! = \sqrt{2\pi n}\, n^n e^{-n}(1 + \omega_n) \qquad (\lim \omega_n = 0)$$

we have

$$\frac{[(2n - 1)!!]^2}{(2n)!} = \frac{(2n)!}{[(2n)!!]^2} = \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1 + \lambda_n}{\sqrt{\pi n}} \qquad (\lim \lambda_n = 0).$$

[8] The proof of this formula can be found in Appendix 1 at the end of the book.

Finally, differentiating the identity

$$\sum_{k=0}^{2n} C_{2n}^k x^k = (1 + x)^{2n}$$

and setting $x = 1$, we find

$$\sum_{k=1}^{2n} kC_{2n}^k = n2^{2n}.$$

Therefore

$$|L_{2n}(0)| \leqq \frac{2n}{2n - 1} \frac{1 + \lambda_n}{\sqrt{\pi n}}, \qquad (75)$$

and hence $\lim_{n\to\infty} L_{2n}(0) = 0$.

Remark. S. N. BERNSTEIN did not investigate the case of $x = 0$. The convergence of the interpolation process in this case was first noted in the thesis of D. L. BERMANN. S. M. LOSINSKI proved that the inequality

$$|L_{2n}(0)| < \frac{A}{n},$$

holds, which is stronger than (75). The LOSINSKI inequality follows from the equation

$$L_{2n}(0) = \left[\frac{(2n - 3)!!}{2^{n-1}(n - 1)!}\right]^2,$$

the proof of which we shall not undertake here.

§3. An Example Due to MARCINKIEWICZ

As we already know, the divergence of an interpolation process occurs because the sequence of functions $\{\lambda_n(x)\}$ is unbounded. When the nodes are equidistant, the inequality

$$|L_{2n+1}(x)| > \frac{q^2}{4n}(1 - x)^{-nx-2} \qquad (x < 0)$$

holds at least for arbitrarily many values of n if not for all such values. If we note that the function $\varphi(x)$ for which we constructed the polynomial $L_{2n+1}(x)$ is in absolute value not greater than one, then clearly for all n the estimate

$$|L_{2n+1}(x)| \leqq \lambda_{2n+1}(x)$$

holds.