The mapping $f \to f^*$ may be viewed as a crude representation, in terms of which phenomena occurring on a general measure space may be viewed in terms of their counterparts on the specific measure space $(0, \infty)$ with Lebesgue measure. Much of the present chapter deals with properties of this representation.

Distribution functions and decreasing rearrangements are introduced in Section 1. Section 2 describes the kinds of measure spaces that admit a reasonable theory of rearrangements. The *maximal function* f^{**} and the *Hardy-Littlewood-Pólya relation* are defined in Section 3. These lead to the introduction of the *rearrangement-invariant spaces*, that is, the Banach function spaces in which functions and their equimeasurable rearrangements have the same norm. The *Luxemburg representation theorem* (Theorem 4.7) allows rearrangement-invariant spaces on more general measure spaces to be studied in the context of the interval $(0, \infty)$ with Lebesgue measure.

Two special families of rearrangement-invariant spaces—the *Lorentz* Λ *and* M *spaces*—are introduced in Section 5. Certain properties of rearrangement-invariant spaces depend only on the values their norms assign to characteristic functions of measurable sets E, that is, on the fundamental function $\varphi(t) = \|\chi_E\|$, where E is a set of measure t, $(0 < t < \infty)$. Modulo one or two minor technicalities, the smallest and the largest of all rearrangement-invariant spaces with given fundamental function φ are the Lorentz spaces $\Lambda(\varphi)$ and $M(\varphi)$, respectively (Corollary 5.14).

The Lorentz spaces $L^1 \cap L^\infty$ and $L^1 + L^\infty$, introduced in Section 6, are respectively the smallest and the largest of all rearrangement-invariant spaces, and play a special role in the theory of interpolation of operators.

Chapter II concludes with a brief discussion of measure-preserving transformations (Section 7). The main result is the theorem of G. G. Lorentz and J. V. Ryff (Theorem 7.5), to the effect that a nonnegative measurable function f on a finite measure space can be obtained from its decreasing rearrangement by composition with a suitable measure-preserving transformation. The converse is false (Example 7.7).

1. DISTRIBUTION FUNCTIONS AND DECREASING REARRANGEMENTS

As in the previous chapter, (R, μ) denotes a totally σ-finite measure space.

Definition 1.1. The *distribution function* μ_f of a function f in $\mathcal{M}_0 = \mathcal{M}_0(R, \mu)$ given by

$$\mu_f(\lambda) = \mu\{x \in R : |f(x)| > \lambda\}, \qquad (\lambda \geq 0). \tag{1.1}$$

Observe that μ_f depends only on the absolute value $|f|$ of the function f, and μ_f may assume the value $+\infty$.

Definition 1.2. Two functions $f \in \mathcal{M}_0(R, \mu)$ and $g \in \mathcal{M}_0(S, \nu)$ are said to be *equimeasurable* if they have the same distribution function, that is, if $\mu_f(\lambda) = \nu_g(\lambda)$ for all $\lambda \geq 0$.

Proposition 1.3. *Suppose f, g, f_n, $(n = 1, 2, \ldots)$, belong to $\mathcal{M}_0(R, \mu)$ and let a be any nonzero scalar. The distribution function μ_f is nonnegative, decreasing, and right-continuous on $[0, \infty)$. Furthermore,*

$$|g| \leq |f| \ \mu\text{-a.e.} \quad \Rightarrow \quad \mu_g \leq \mu_f; \tag{1.2}$$

$$\mu_{af}(\lambda) = \mu_f(\lambda/|a|), \qquad (\lambda \geq 0); \tag{1.3}$$

$$\mu_{f+g}(\lambda_1 + \lambda_2) \leq \mu_f(\lambda_1) + \mu_g(\lambda_2), \qquad (\lambda_1, \lambda_2 \geq 0); \tag{1.4}$$

$$|f| \leq \liminf_{n\to\infty} |f_n| \ \mu\text{-a.e.} \quad \Rightarrow \quad \mu_f \leq \liminf_{n\to\infty} \mu_{f_n}; \tag{1.5}$$

in particular,

$$|f_n| \uparrow |f| \ \mu\text{-a.e.} \quad \Rightarrow \quad \mu_{f_n} \uparrow \mu_f.$$

Proof. It is clear that μ_f is nonnegative and decreasing (in the wide sense: $0 \leq \lambda_1 < \lambda_2 \Rightarrow \mu_f(\lambda_1) \geq \mu_f(\lambda_2)$). To establish right-continuity, let $E(\lambda) = \{x : |f(x)| > \lambda\}$, $(\lambda \geq 0)$, and fix $\lambda_0 \geq 0$. The sets $E(\lambda)$ increase as λ decreases, and

$$E(\lambda_0) = \bigcup_{\lambda > \lambda_0} E(\lambda) = \bigcup_{n=1}^{\infty} E\left(\lambda_0 + \frac{1}{n}\right).$$

Hence, by the monotone convergence theorem,

$$\mu_f\left(\lambda_0 + \frac{1}{n}\right) = \mu\left(E\left(\lambda_0 + \frac{1}{n}\right)\right) \uparrow \mu(E(\lambda_0)) = \mu_f(\lambda_0),$$

and this establishes the right-continuity.

Properties (1.2) and (1.3) are immediate consequences of Definition 1.1. Property (1.4) follows from the obvious fact that if $|f(x) + g(x)| > \lambda_1 + \lambda_2$, then either $|f(x)| > \lambda_1$ or $|g(x)| > \lambda_2$. To establish (1.5), fix $\lambda \geq 0$ and let

$$E = \{x : |f(x)| > \lambda\}, \qquad E_n = \{x : |f_n(x)| > \lambda\}, \qquad (n = 1, 2, \ldots).$$

Clearly, $E \subset \bigcup_{m=1}^{\infty} \bigcap_{n>m} E_n$. Hence,

$$\mu\left(\bigcap_{n>m} E_n\right) \leq \inf_{n>m} \mu(E_n) \leq \sup_m \inf_{n>m} \mu(E_n) = \liminf_{n\to\infty} \mu(E_n)$$

or each $m = 1, 2, \ldots$. But $\bigcap_{n>m} E_n$ increases with m, so an appeal to the nonotone convergence theorem gives

$$\mu(E) \leq \mu\left(\bigcup_{m=1}^{\infty} \bigcap_{n>m} E_n\right) = \lim_{m\to\infty} \mu\left(\bigcap_{n>m} E_n\right) \leq \liminf_{n\to\infty} \mu(E_n).$$

This establishes the first of the assertions in (1.5). The second is an immediate onsequence of the first. ■

Example 1.4. It will be worthwhile to formally compute the distribution inction μ_f of a nonnegative simple function f. Suppose

$$f(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x), \tag{1.6}$$

here the sets E_j are pairwise disjoint subsets of R with finite μ-measure nd $a_1 > a_2 > \cdots > a_n > 0$. If $\lambda \geq a_1$, then clearly $\mu_f(\lambda) = 0$. However, if $_2 \leq \lambda < a_1$, then $f(x)$ exceeds λ precisely on the set E_1, and so $\mu_f(\lambda) =$ (E_1). Similarly, if $a_3 \leq \lambda < a_2$, then $f(x)$ exceeds λ precisely on $E_1 \cup E_2$, nd so $\mu_f(\lambda) = \mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$. In general, we have

$$\mu_f(\lambda) = \sum_{j=1}^{n} m_j \chi_{[a_{j+1}, a_j)}(\lambda), \qquad (\lambda \geq 0), \tag{1.7}$$

here

$$m_j = \sum_{i=1}^{j} \mu(E_i), \qquad (j = 1, 2, \ldots, n) \tag{1.8}$$

nd a_{n+1} is defined to be zero (see Figure 1).

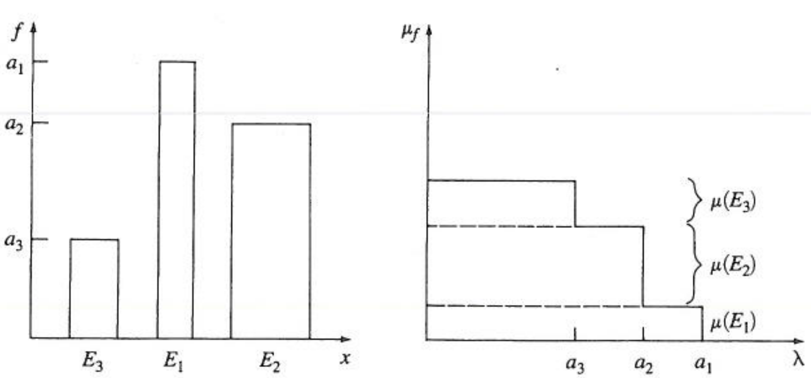

Figure 1. Graphs of $f(x)$ and $\mu_f(\lambda)$.

Definition 1.5. Suppose f belongs to $\mathcal{M}_0(R, \mu)$. The *decreasing rearrangement* of f is the function f^* defined on $[0, \infty)$ by

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \leq t\}, \qquad (t \geq 0). \tag{1.9}$$

We use here the convention that $\inf \varnothing = \infty$. Thus, if $\mu_f(\lambda) > t$ for all $\lambda \geq 0$, then $f^*(t) = \infty$. Also, if (R, μ) is a finite measure space, then the distribution function μ_f is bounded by $\mu(R)$ and so $f^*(t) = 0$ for all $t \geq \mu(R)$. In this case, we may regard f^* as a function defined on the interval $[0, \mu(R))$. Notice also that if μ_f happens to be continuous and strictly decreasing, then f^* is simply the inverse of μ_f on the appropriate interval. In fact, for general f, if we first form the distribution function μ_f and then form the distribution function m_{μ_f} of μ_f (with respect to Lebesgue measure m on $[0, \infty)$) we obtain precisely the decreasing rearrangement f^*. This is an immediate consequence of the identities

$$f^*(t) = \sup\{\lambda : \mu_f(\lambda) > t\} = m_{\mu_f}(t), \qquad (t \geq 0), \tag{1.10}$$

which follow from (1.9), the fact that μ_f is decreasing, and the definition of the distribution function.

Examples 1.6. (a) Now we compute the decreasing rearrangement of the simple function f given by (1.6). Referring to (1.9) and Figure 1, we see that $f^*(t) = 0$ if $t \geq m_3$. Also, if $m_3 > t \geq m_2$, then $f^*(t) = a_3$, and if $m_2 > t \geq m_1$, then $f^*(t) = a_2$, and so on. Hence,

$$f^*(t) = \sum_{j=1}^{n} a_j \chi_{[m_{j-1}, m_j)}(t), \qquad (t \geq 0), \tag{1.11}$$

where we have taken $m_0 = 0$.

Geometrically, we are merely rearranging the *vertical* blocks in the graph of f in decreasing order to obtain the decreasing rearrangement f^* (see Figure 2); the values of f^* at the jumps are determined by the right continuity (Proposition 1.7).

(b) It is sometimes more useful to section functions into *horizontal* blocks rather than vertical ones. Thus, the simple function f in (1.6) may be represented also as follows:

$$f(x) = \sum_{k=1}^{n} b_k \chi_{F_k}(x), \tag{1.12}$$

where the coefficients b_k are positive and the sets F_k each have finite measure and form an increasing sequence $F_1 \subset F_2 \subset \cdots \subset F_n$. Comparison with (1.6)

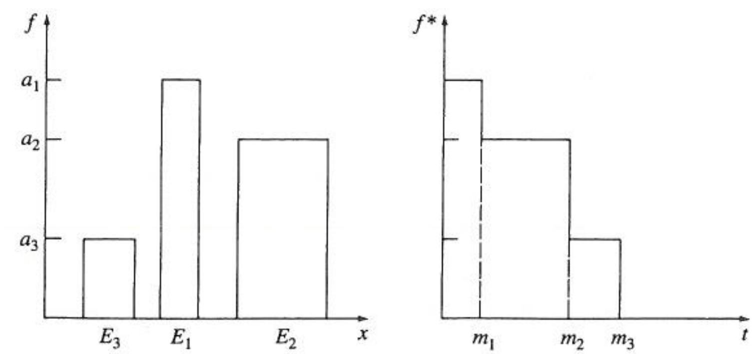

Figure 2. Graphs of f and f^*.

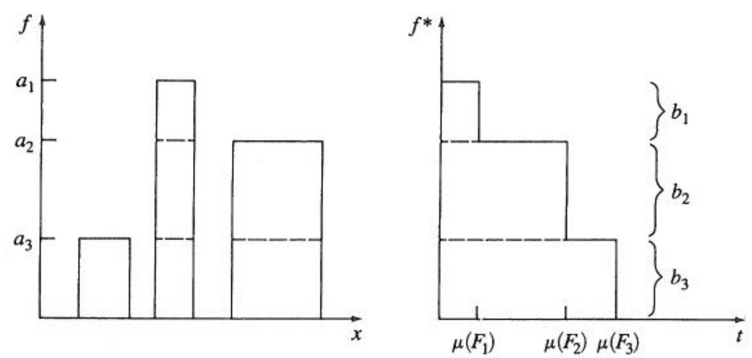

Figure 3. Graphs of f and f^*.

shows that

$$b_k = a_k - a_{k+1}, \qquad F_k = \bigcup_{j=1}^{k} E_j, \qquad (k = 1, 2, \ldots, n).$$

In this case, the decreasing rearrangement is viewed as being formed by sliding the blocks in each horizontal layer to form a single larger block positioned with its left-hand end against the vertical axis (see Figure 3). Thus

$$f^* = \sum_{k=1}^{n} b_k \chi_{[0, \mu(F_k))}. \tag{1.13}$$

(c) Let $f(x) = 1 - e^{-x}$, $(0 < x < \infty)$. The distribution function m_f (with respect to Lebesgue measure m on $(0, \infty)$) is infinite for $0 \leq \lambda < 1$, and equal to

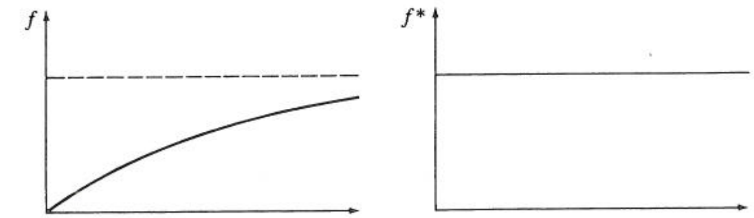

Figure 4. Graphs of $f(x) = 1 - e^{-x}$ and $f^*(t)$.

zero for all $\lambda \geq 1$. Hence $f^*(t) = 1$ for all $t \geq 0$ (cf. Figure 4). This example shows that a considerable amount of information may be lost in passing to the decreasing rearrangement. Such information, however, is irrelevant as far as L^p-norms (or any other rearrangement-invariant norms) are concerned. Thus, the L^p-norms of f and f^* are both infinite when $1 \leq p < \infty$, and the L^∞-norms are both equal to 1.

Proposition 1.7. *Suppose $f, g,$ and f_n, $(n = 1, 2, \ldots)$, belong to $\mathcal{M}_0(R, \mu)$ and let a be any scalar. The decreasing rearrangement f^* is a nonnegative, decreasing, right-continuous function on $[0, \infty)$. Furthermore,*

$$|g| \leq |f| \ \mu\text{-a.e.} \quad \Rightarrow \quad g^* \leq f^*; \tag{1.14}$$

$$(af)^* = |a| f^*; \tag{1.15}$$

$$(f + g)^*(t_1 + t_2) \leq f^*(t_1) + g^*(t_2), \qquad (t_1, t_2 \geq 0); \tag{1.16}$$

$$|f| \leq \liminf_{n \to \infty} |f_n| \ \mu\text{-a.e.} \quad \Rightarrow \quad f^* \leq \liminf_{n \to \infty} f_n^*; \tag{1.17}$$

in particular,

$$|f_n| \uparrow |f| \ \mu\text{-a.e.} \quad \Rightarrow \quad f_n^* \uparrow f^*;$$

$$f^*(\mu_f(\lambda)) \leq \lambda, \qquad (\mu_f(\lambda) < \infty); \qquad \mu_f(f^*(t)) \leq t, \qquad (f^*(t) < \infty); \tag{1.18}$$

$$f \text{ and } f^* \text{ are equimeasurable;} \tag{1.19}$$

$$(|f|^p)^* = (f^*)^p, \qquad (0 < p < \infty). \tag{1.20}$$

Proof. That f^* is nonnegative, decreasing, and right-continuous follows from Proposition 1.3 and the fact that f^* is itself a distribution function (cf. (1.10)). The properties (1.14), (1.15), and (1.17) are immediate consequences of their counterparts in Proposition 1.3 and the definition of the decreasing rearrangement.

For property (1.18), fix $\lambda \geq 0$ and suppose $t = \mu_f(\lambda)$ is finite. Then (1.9) gives

$$f^*(\mu_f(\lambda)) = f^*(t) = \inf\{\lambda' : \mu_f(\lambda') \leq t = \mu_f(\lambda)\} \leq \lambda,$$

which establishes the first part of (1.18). For the second part, fix $t \geq 0$ and uppose $\lambda = f^*(t)$ is finite. By (1.9), there is a sequence $\lambda_n \downarrow \lambda$ with $\mu_f(\lambda_n) \leq t$, so he right-continuity of μ_f (Proposition 1.3) gives

$$\mu_f(f^*(t)) = \mu_f(\lambda) = \lim_{n\to\infty} \mu_f(\lambda_n) \leq t.$$

'his establishes (1.18).

Returning to (1.16), we may assume that $\lambda = f^*(t_1) + g^*(t_2)$ is finite since therwise there is nothing to prove. Let $t = \mu_{f+g}(\lambda)$. Then by the triangle ıequality and the second of the inequalities in (1.18) we have

$$\begin{aligned} t &= \mu\{x : |f(x) + g(x)| > f^*(t_1) + g^*(t_2)\} \\ &\leq \mu\{x : |f(x)| > f^*(t_1)\} + \mu\{x : |g(x)| > g^*(t_2)\} \\ &= \mu_f(f^*(t_1)) + \mu_g(g^*(t_2)) \\ &\leq t_1 + t_2. \end{aligned}$$

his shows in particular that t is finite. Hence, using the first of the inequalities ı (1.18) and the fact that $(f+g)^*$ is decreasing, we obtain

$$\begin{aligned} (f+g)^*(t_1+t_2) &\leq (f+g)^*(t) = (f+g)^*(\mu_{f+g}(\lambda)) \\ &\leq \lambda = f^*(t_1) + g^*(t_2), \end{aligned}$$

ıd this establishes (1.16).

For an arbitrary function f in $\mathcal{M}_0$, we can find a sequence of nonnegative mple functions f_n, $(n = 1, 2, \ldots)$, such that $f_n \uparrow |f|$. It is clear (cf. Examıe 1.6(a)) that for each n the functions f_n and f_n^* are equimeasurable, that is,

$$\mu_{f_n}(\lambda) = m_{f_n^*}(\lambda), \qquad (\lambda \geq 0). \tag{1.21}$$

ut $f_n \uparrow |f|$ and $f_n^* \uparrow f^*$ (by 1.17) so property (1.5), applied to each of the stribution functions in (1.21), shows that

$$\mu_f(\lambda) = m_{f^*}(\lambda), \qquad (\lambda \geq 0). \tag{1.22}$$

ence, f and f^* are equimeasurable, as asserted by (1.19).

Finally, from (1.22) we have

$$\mu_{|f|^p}(\lambda) = \mu_f(\lambda^{1/p}) = m_{f^*}(\lambda^{1/p}) = m_{(f^*)^p}(\lambda), \qquad (\lambda \geq 0).$$

assing to the decreasing rearrangements by means of (1.9), we obtain .20). ■

The next result gives alternative descriptions of the L^p-norm in terms of the distribution function and the decreasing rearrangement.

Proposition 1.8. *Let $f \in \mathcal{M}_0$. If $0 < p < \infty$, then*

$$\int_R |f|^p \, d\mu = p \int_0^\infty \lambda^{p-1} \mu_f(\lambda)\, d\lambda = \int_0^\infty f^*(t)^p \, dt. \tag{1.23}$$

Furthermore, in the case $p = \infty$,

$$\operatorname*{ess\,sup}_{x \in R} |f(x)| = \inf\{\lambda : \mu_f(\lambda) = 0\} = f^*(0). \tag{1.24}$$

Proof. In view of (1.5), (1.17), and the monotone convergence theorem, it will suffice to prove (1.23) for an arbitrary nonnegative simple function f. With f written in the form (1.6), we saw that its decreasing rearrangement f^* is given by (1.11). But then it is clear from (1.8) that

$$\int |f|^p \, d\mu = \sum_{j=1}^n a_j^p \mu(E_j) = \sum_{j=1}^n a_j^p m([m_{j-1}, m_j)) = \int_0^\infty (f^*)^p \, dm.$$

Similarly, using the expressions (1.6) and (1.7) for f and its distribution function μ_f, we have

$$\begin{aligned} p \int_0^\infty \lambda^{p-1} \mu_f(\lambda)\, d\lambda &= p \sum_{j=1}^n m_j \int_{a_{j+1}}^{a_j} \lambda^{p-1}\, d\lambda = \sum_{j=1}^n (a_j^p - a_{j+1}^p) m_j \\ &= \sum_{j=1}^n a_j^p \mu(E_j) = \int |f|^p \, d\mu, \end{aligned}$$

where the third equality follows from (1.8) and a summation by parts.

This establishes (1.23). The proof of (1.24) is straightforward and we omit it. ■

2. AN INEQUALITY OF HARDY AND LITTLEWOOD

While the decreasing rearrangement does not necessarily preserve sums or products of functions, there are nevertheless some basic inequalities that govern these processes. These will be derived in the present section (for products) and in the next section (for sums).

The starting point is an elementary inequality due to G. H. Hardy and J. E. Littlewood. The inequality involves finite sequences $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ of nonnegative real numbers, and asserts that

$$\sum_{j=1}^n a_j b_j \leq \sum_{j=1}^n a_j^* b_j^*, \tag{2.1}$$

where $(a_j^*)_{j=1}^n$ denotes the sequence of elements a_j arranged in decreasing order (and similarly for $(b_j^*)_{j=1}^n$). In other words, the sum attains its maximum value when the terms in each sequence are rearranged in decreasing order.

It will often be convenient to regard such a sequence $(a_j)_{j=1}^n$ as a simple function $f = \sum_{j=1}^n a_j \chi_{[j-1,j)}$ defined on the interval $[0, \infty)$. In that case, as Example 1.6(a) shows, the decreasing rearrangement f^* of f is just the simple function $f^* = \sum_{j=1}^n a_j^* \chi_{[j-1,j)}$ corresponding to the rearranged sequence $(a_j^*)_{j=1}^n$. Viewed in this way, the inequality (2.1) will be seen as a special case of the more general integral inequality (2.3) to be established in Theorem 2.2 below.

Lemma 2.1. *Let g be a nonnegative simple function on (R, μ) and let E be an arbitrary μ-measurable subset of R. Then*

$$\int_E g\, d\mu \le \int_0^{\mu(E)} g^*(s)\, ds. \tag{2.2}$$

Proof. We express g in the form of (1.12). Thus

$$g(x) = \sum_{j=1}^n b_j \chi_{F_j}(x),$$

where $F_1 \subset F_2 \subset \cdots \subset F_n$ and $b_j > 0$, $(j = 1, 2, \ldots, n)$. Using the formulation (1.13) for the decreasing rearrangement, we obtain

$$\int_E g\, d\mu = \sum_{j=1}^n b_j \mu(E \cap F_j) \le \sum_{j=1}^n b_j \cdot \min(\mu(E), \mu(F_j))$$

$$= \sum_{j=1}^n b_j \int_0^{\mu(E)} \chi_{(0,\mu(F_j))}(s)\, ds = \int_0^{\mu(E)} g^*(s)\, ds. \quad \blacksquare$$

Theorem 2.2. **(G. H. Hardy & J. E. Littlewood).** *If f and g belong to $\mathcal{M}_0 = \mathcal{M}_0(R, \mu)$, then*

$$\int_R |fg|\, d\mu \le \int_0^\infty f^*(s) g^*(s)\, ds. \tag{2.3}$$

Proof. Since f^* and g^* depend only on the absolute values of f and g, it is enough to establish (2.3) for nonnegative functions f and g. But then, in view of (1.17) and the monotone convergence theorem, there is no loss of generality in supposing f and g to be simple. In that case, we may write

$$f(x) = \sum_{j=1}^m a_j \chi_{E_j}(x),$$

where $E_1 \subset E_2 \subset \cdots \subset E_m$ and $a_j > 0$, $(j = 1, 2, \ldots, m)$ (cf. (1.12)). Then (cf. (1.13))

$$f^*(t) = \sum_{j=1}^m a_j \chi_{[0,\mu(E_j))}(t).$$

Hence, by Lemma 2.1,

$$\int |fg|\, d\mu = \sum_{j=1}^m a_j \int_{E_j} g\, d\mu \le \sum_{j=1}^m a_j \int_0^{\mu(E_j)} g^*(s)\, ds$$

$$= \int_0^\infty \sum_{j=1}^m a_j \chi_{[0,\mu(E_j))}(s) g^*(s)\, ds$$

$$= \int_0^\infty f^*(s) g^*(s)\, ds,$$

as desired. $\blacksquare$

An immediate consequence of the Hardy-Littlewood inequality (2.3) is that

$$\int_R |f\tilde{g}|\, d\mu \le \int_0^\infty f^*(t) g^*(t)\, dt \tag{2.4}$$

for every function $\tilde{g}$ on R equimeasurable with g. When f and g are finite sequences, as in (2.1), it is clear that equality is attained in (2.4) for a suitable $\tilde{g}$. Indeed, such a $\tilde{g}$ may evidently be obtained from g by means of a suitable permutation of the underlying point-set $\{1, 2, \ldots, n\}$. The situation is not as straightforward, however, for more general measure spaces and so it becomes necessary to isolate those classes of measure spaces with the desired properties.

Much of theory can be developed under a weaker requirement than the attainment of equality in (2.4). What is needed is that the *supremum* of the integrals on the left of (2.4) coincide with the value on the right; such measure spaces will be called *resonant*. If the supremum is in fact attained, then the measure space will be called *strongly resonant* (cf. Definition 2.3 below).

Naturally, we shall need to know which measure spaces possess these resonance properties. We shall see (Theorem 2.7) that a totally σ-finite measure space is resonant if and only if it is nonatomic or completely atomic with all atoms having equal measure. The strongly resonant measure spaces are precisely the resonant ones of finite measure (Theorem 2.6).

Definition 2.3. A totally σ-finite measure space (R, μ), is said to be *resonant* if, for each f and g in $\mathcal{M}_0(R, \mu)$, the identity

$$\int_0^\infty f^*(t) g^*(t)\, dt = \sup \int_R |f\tilde{g}|\, d\mu \tag{2.5}$$

holds, where the supremum is taken over all functions $\tilde{g}$ on R equimeasurable with g.

Similarly, (R, μ) is said to be *strongly resonant* if, for each pair of functions f and g in $\mathcal{M}_0(R, \mu)$, there exists a function $\tilde{g}$ on R equimeasurable with g such that

$$\int_0^\infty f^*(t)g^*(t)\,dt = \int_R |f\tilde{g}|\,d\mu. \tag{2.6}$$

Examples **2.4.** (a) It is clear from (2.4) that every strongly resonant measure space is resonant. The converse, however, is false. Indeed, we shall see in Theorem 2.7 that the interval $[0, \infty)$ with Lebesgue measure is resonant, but the following example shows that it is not strongly resonant. Let f be the function described in Example 1.6(c). Thus, $f(x) = 1 - e^{-x}$, $(0 \le x < \infty)$, and $f^* \equiv 1$ (see Figure 4). Let $g = g^* = \chi_{[0,1)}$. If $\tilde{g}$ is equimeasurable with g, then $|\tilde{g}|$ is the characteristic function of some subset E of $(0, \infty)$ of measure 1. But then, as is evident from Figure 4,

$$\int_R |f\tilde{g}|\,d\mu = \int_E (1 - e^{-x})\,dx < 1 = \int_0^\infty f^*(t)g^*(t)\,dt,$$

so equality is never attained.

(b) Here is an example of a measure space that is not resonant. Let (R, μ) consist of two atoms a and b with $\mu(a) = 1$ and $\mu(b) = 2$. Let $f = \chi_b$ and $g = \chi_a$. Then $f^* = \chi_{[0,2)}$ and $g^* = \chi_{[0,1)}$ so $\int f^*g^* = 1$. But $\int |f\tilde{g}|\,d\mu = 0$ for every $\tilde{g}$ equimeasurable with g because f and $\tilde{g}$ must have disjoint supports. This argument shows in fact that no measure space containing two atoms with positive but unequal measures can be resonant.

Lemma 2.5. *Let (R, μ) be a finite nonatomic measure space. Suppose f belongs to $\mathcal{M}_0(R, \mu)$ and let t be any number satisfying $0 \le t \le \mu(R)$. Then there is a measurable set E_t, with $\mu(E_t) = t$, such that*

$$\int_{E_t} |f|\,d\mu = \int_0^t f^*(s)\,ds. \tag{2.7}$$

Moreover, the sets E_t can be constructed so as to increase with t:

$$0 \le s \le t \le \mu(R) \quad \Rightarrow \quad E_s \subset E_t. \tag{2.8}$$

Proof. We consider two separate cases, according to whether the number t does or does not lie in the range of the distribution function μ_f of f.

Suppose first that there exists $\alpha > 0$ for which $\mu_f(\alpha) = t$. In that case, it follows from (1.9) that

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) = t\},$$

and then the right-continuity of μ_f gives $\mu_f(f^*(t)) = t$. Equivalently, the set $E_t = \{x : |f(x)| > f^*(t)\}$ has measure $\mu(E_t) = t$, and this shows that the distribution function of $f\chi_{E_t}$ is given by

$$\mu\{x \in E_t : |f(x)| > \lambda\} = \begin{cases} \mu_f(\lambda), & \lambda > f^*(t), \\ t, & 0 \le \lambda \le f^*(t). \end{cases}$$

Also, the distribution function of $f^*\chi_{[0,t)}$ is given by

$$m\{s \in [0, t] : f^*(s) > \lambda\} = \begin{cases} m_{f^*}(\lambda), & \lambda > f^*(t) \\ t, & 0 \le \lambda \le f^*(t). \end{cases}$$

But f and f^* are equimeasurable, so $\mu_f = m_{f^*}$. Hence, $f\chi_{E_t}$ and $f^*\chi_{[0,t)}$ are equimeasurable. In particular, by Proposition 1.8, their integrals are the same:

$$\int_{E_t} |f|\,d\mu = \int_R |f\chi_{E_t}|\,d\mu = \int_0^\infty f^*(s)\chi_{[0,t]}(s)\,ds = \int_0^t f^*(s)\,ds.$$

Hence, (2.7) holds. Notice that the sets E_t increase with t.

Next we consider the case where t is not in the range of μ_f (cf. Figure 5). Observe that since $\mu(R)$ is finite and $f \in \mathcal{M}_0$, the dominated convergence theorem gives

$$0 = \mu\{x : |f(x)| = \infty\} = \lim_{n\to\infty} \mu\{x : |f(x)| > n\} = \lim_{n\to\infty} \mu_f(n).$$

Hence, since μ_f is decreasing,

$$\lim_{\lambda\to\infty} \mu_f(\lambda) = 0. \tag{2.9}$$

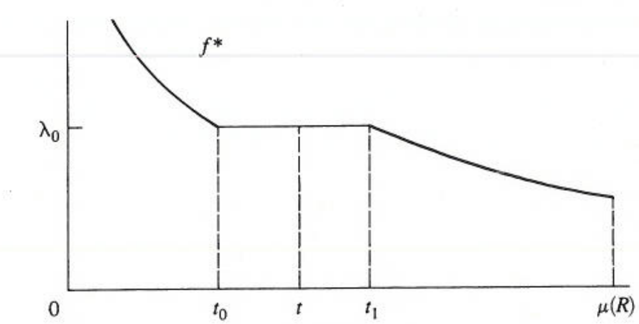

Figure 5

Let $\lambda_0 = f^*(t)$ and suppose first that $\lambda_0 > 0$. Since $t > 0$, it follows from (1.9) and (2.9) that λ_0 is finite. Also, since t is not in the range of μ_f, we see from (1.18) and (1.9) that $\mu_f(\lambda_0) < t < \mu_f(\lambda)$ for $0 < \lambda < \lambda_0$. Thus, if t_1 denotes the left-hand limit $\mu_f(\lambda_0-)$ (which exists because μ_f is decreasing), then

$$t_0 \equiv \mu_f(\lambda_0) < t \leq \mu_f(\lambda_0-) = t_1. \tag{2.10}$$

This, together with (1.9), shows in fact that

$$f^*(s) = \lambda_0, \qquad (t_0 \leq s < t_1) \tag{2.11}$$

(cf. Figure 5).

We claim that

$$t_1 = \mu\{x : |f(x)| \geq \lambda_0\}. \tag{2.12}$$

To see this, we need only express the set on the right as the intersection of the decreasing sequence of sets

$$F_n = \left\{x : |f(x)| > \lambda_0 - \frac{1}{n}\right\}, \qquad (n = 1, 2, \ldots),$$

observe that $\mu(R)$ is finite, and use the dominated convergence theorem to obtain

$$\mu\{x : |f(x)| \geq \lambda_0\} = \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} \mu_f\left(\lambda_0 - \frac{1}{n}\right) = \mu_f(\lambda_0-).$$

It follows from (2.10) and (2.12) that the set $G = \{x : |f(x)| = \lambda_0\}$ has measure $t_1 - t_0$. Since (R, μ) is nonatomic, we may thus select a subset F_t of G with measure $t - t_0$ (notice that there is considerable freedom in this choice if $t < t_1$). The set E_t defined by the disjoint union

$$E_t = \{x : |f(x)| > \lambda_0\} \cup F_t \tag{2.13}$$

then has measure $\mu_f(\lambda_0) + (t - t_0) = t$, as desired. Moreover,

$$\int_{E_t} |f|\, d\mu = \int_{\{|f| > \lambda_0\}} |f|\, d\mu + \int_{F_t} |f|\, d\mu \tag{2.14}$$

Now, by (2.10), the value t_0 does lie in the range of μ_f. The first part of the proof shows therefore that the first of the integrals on the right of (2.14) has value $\int_0^{t_0} f^*(s)\, ds$. Also, since $|f| = \lambda_0$ on F_t, the second integral has value

$$\lambda_0 \mu(F_t) = \lambda_0(t - t_0) = \int_{t_0}^{t} f^*(s)\, ds,$$

by (2.11). Combining these results with (2.14), we obtain (2.7).

In the remaining case where $\lambda_0 = 0$, we obtain in place of (2.10) the inequality $\mu\{x : |f(x)| > 0\} \equiv t_0 < t$. In that case, we may choose F_t disjoint from the support of f, with measure $\mu(F_t) = t - t_0$. Defining E_t again by (2.13), and observing that $f^*(s)$ vanishes for $s \geq t_0$, we obtain (2.7) as before:

$$\int_{E_t} |f|\, d\mu = \int_0^{t_0} f^*(s)\, ds = \int_0^{t} f^*(s)\, ds.$$

We observed in the first part of the proof that the sets E_t satisfy (2.8) for all those values of t in the range of μ_f. For the other values of t, it is elementary (cf. Exercise 17) to show that the sets F_t in the second part of the proof may be chosen so as to increase with t (in each of the intervals $(t_0, t_1]$ in the complement of the range of μ_f). Given this construction, a moment's reflection will now show that (2.8) holds as desired. ■

We are now in a position to characterize the strongly resonant measure spaces (Theorem 2.6) and the resonant measure spaces (Theorem 2.7).

Theorem 2.6. *A totally σ-finite measure space (R, μ) is strongly resonant if and only if it is a finite measure space of one of the following two types:*

(i) *nonatomic;*
(ii) *completely atomic, with all atoms having equal measure.*

Proof. First, we establish the sufficiency. Thus, given f and g in $\mathcal{M}_0(R, \mu)$, we need to find a function $\tilde{g}$ equimeasurable with g such that

$$\int_R |f\tilde{g}|\, d\mu = \int_0^{\infty} f^*(t) g^*(t)\, dt. \tag{2.15}$$

It is clear that we may assume f and g nonnegative.

If (R, μ) has finite measure and is of type (ii), then a suitable $\tilde{g}$ may be constructed from g merely by permuting the underlying (finitely many) atoms. Indeed, the desired permutation is just the composition of the inverse of the permutation that takes f to f^* with the permutation that takes g to g^*. Hence, we need only consider the case where (R, μ) is nonatomic and has finite measure.

Let $(g_n)_{n=1}^{\infty}$ be an increasing sequence of nonnegative simple functions such that $g_n \uparrow g$ μ-a.e. We shall construct an increasing sequence $(\tilde{g}_n)_{n=1}^{\infty}$ of nonnegative simple functions such that $\tilde{g}_n$ is equimeasurable with g_n and

$$\int_R |f\tilde{g}_n|\, d\mu = \int_0^{\infty} f^*(t)\tilde{g}_n^*(t)\, dt, \qquad (n = 1, 2, \ldots). \tag{2.16}$$

hen

$$f = (\operatorname{sgn} g)S(f \cdot \overline{\operatorname{sgn} f}) = (\operatorname{sgn} g)S|f| = (\operatorname{sgn} g)|g| = g,$$

 $Tf = g$, as desired. Since S is a contraction on L^1 and on L^∞, the same is idently true of T. Hence, T is admissible and

$$\|T\|_{\mathscr{A}} = \max\{\|T\|_{\mathscr{B}(L^1)}, \|T\|_{\mathscr{B}(L^\infty)}\} \leq 1. \quad ■$$

We are now in a position to establish the main result of this section.

heorem 2.12. (A. P. Calderón). *Let X be a Banach function space over a sonant measure space. Then X is an exact interpolation space between L^1 and $^\infty$ if and only if X is rearrangement-invariant.*

oof. We have seen already in Theorem 2.2 that X is an exact interpolation ace if it is rearrangement-invariant. Suppose conversely that X is an exact terpolation space. To establish rearrangement-invariance, we need to show at if f is an arbitrary function in X, and if g is equimeasurable with f, $\sim f$), then g belongs to X and $\|g\|_X = \|f\|_X$. But if $g \sim f$, then certainly $\prec f$ and so, by Corollary 2.11, there is an admissible operator T with $\|_{\mathscr{A}} \leq 1$ such that $Tf = g$. Since X is an exact interpolation space, the func- n $g = Tf$ belongs to X, and from (1.16) we have

$$\|g\|_X = \|Tf\|_X \leq \|T\|_{\mathscr{B}(X)}\|f\|_X \leq \|T\|_{\mathscr{A}}\|f\|_X \leq \|f\|_X.$$

nce f and g are equimeasurable, we also have $f \prec g$ and so we may repeat e above argument with the roles of f and g interchanged to obtain $\|_X \leq \|g\|_X$. Hence, $\|f\|_X = \|g\|_X$ and the proof is complete. ■

Theorem 2.12 characterizes the Banach function spaces that are inter- lation spaces between L^1 and L^∞ but, as it stands, does not characterize all erpolation spaces between L^1 and L^∞. Little modification is needed wever to achieve such a characterization and this will be done in Section 1 of apter V (see Theorem V.1.17).

It is easy to check that the operator T constructed in the proof of Theo- n 2.10 also satisfies property (2.10), that is, $|Th| \leq T|h|$ μ-a.e. for every h $L^1 + L^\infty$. We shall not make use of this property here.

However, we shall make use of the observation that Corollary 2.11 remains id when f and g are defined on *different* measure spaces. The proof carries er almost verbatim. For the application we have in mind, the function g will the decreasing rearrangement f^* of f, so the hypothesis $g \prec f$ of Corol- y 2.11 is trivially satisfied. The result we shall need is then as follows.

Corollary 2.13. *Let (R, μ) be a resonant measure space and let $\tilde{R}$ denote the interval $(0, \mu(R))$ equipped with Lebesgue measure m. Let f belong to $(L^1 + L^\infty)(R, \mu)$. Then there is an admissible operator T, with respect to the couples $(L^1, L^\infty)(R, \mu)$ and $(L^1, L^\infty)(\tilde{R}, m)$, of norm at most 1, such that $Tf = f^*$ m-a.e. on $\tilde{R}$.*

3. THE HARDY-LITTLEWOOD MAXIMAL OPERATOR

The L^p-boundedness hypothesized in Theorem 2.2 ($p = 1$ and $p = \infty$) is too strong for many potential applications and our goal in the remainder of this chapter will be to develop an interpolation theory under weaker hypotheses. The basis of this so-called *weak-type interpolation theory* is a fundamental interpolation theorem due to J. Marcinkiewicz. Before presenting the Marcinkiewicz interpolation theorem, however, it will be instructive to consider some examples of operators for which the preceding interpolation theory is inadequate and upon which the weak-type theory will be founded. We begin in this section with the Hardy-Littlewood maximal operator, and continue in the next section with a discussion of the Hilbert transform.

Definition 3.1. Let f be a locally integrable function on $\mathbf{R}^n$. The *Hardy-Littlewood maximal function Mf* of f is defined by

$$(Mf)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy, \qquad (x \in \mathbf{R}^n), \tag{3.1}$$

where the supremum extends over all cubes Q containing x (here, as throughout, cubes will be assumed to have their sides parallel to the coordinate axes). The operator $M: f \to Mf$ is called the *Hardy-Littlewood maximal operator*.

Observe that M is merely sublinear, rather than linear:

$$M(f + g) \leq Mf + Mg; \qquad M(\lambda f) = |\lambda| Mf.$$

It is clear that M is a contraction on L^∞:

$$\|Mf\|_{L^\infty} \leq \|f\|_{L^\infty}, \qquad (f \in L^\infty), \tag{3.2}$$

but not on L^1. In fact, it is easy to see that $(Mf)(x)$ never decays faster than $|x|^{-n}$ as $|x| \to \infty$, that is,

$$(Mf)(x) \geq \frac{c}{|x|^n}, \qquad (|x| > 1),$$

f f is nonzero on a set of positive measure. In particular, Mf is *never* ntegrable unless $f = 0$ a.e. In fact, Mf need not even be locally integrable, as he one-dimensional example

$$f(x) = \begin{cases} \dfrac{1}{|x|(\log|x|)^2}, & |x| \le \dfrac{1}{2}, \\ 0, & |x| > \dfrac{1}{2} \end{cases}$$

hows.

Our first objective, an estimate for the size of Mf for integrable f, will equire the following elementary variant of the Vitali covering lemma.

.emma 3.2. *Let Ω be an arbitrary measurable subset of $\boldsymbol{R}^n$ of finite measure. .et $\mathscr{F}$ be a collection of cubes Q that covers Ω. Then there exist finitely many isjoint cubes, say $Q_1, Q_2, \ldots, Q_K$, from $\mathscr{F}$ such that*

$$\sum_{k=1}^{K} |Q_k| \ge 4^{-n}|\Omega|. \tag{3.3}$$

'roof. By the inner regularity of Lebesgue measure on $\boldsymbol{R}^n$, the set Ω can be pproximated arbitrarily closely from within by compact sets. Furthermore, ach cube Q in $\mathscr{F}$ can be replaced by a larger open cube with the ratio of the ieasures arbitrarily close to 1. Hence, it will suffice to establish the lemma nder the assumptions that Ω is compact and each cube Q in $\mathscr{F}$ is open, :ovided that (3.3) is then achieved with a constant larger than 4^{-n} (our proof ill give constant 3^{-n}, which will suffice).

With these hypotheses, $\mathscr{F}$ is an open cover of the compact set Ω so there :ist finitely many cubes in $\mathscr{F}$ which cover Ω. Hence, we may as well assume so that $\mathscr{F}$ is finite. Let Q_1 be the largest of the cubes in $\mathscr{F}$, let Q_2 be the largest maining cube in $\mathscr{F}$ disjoint from Q_1, let Q_3 be the largest disjoint from Q_1 ıd Q_2, and so on. Since $\mathscr{F}$ is finite, this process ends after finitely many steps, elding disjoint cubes $Q_1, Q_2, \ldots Q_K$, say. Let $\bar{Q}_k$ be the cube concentric with $_k$ but with side-length three times as large, $(k = 1, 2, \ldots, K)$. We claim that e cubes $\bar{Q}_k$ cover Ω. For, if not, then there is a point x in $\Omega \setminus \bigcup_{k=1}^{K} \bar{Q}_k$ and, ıce $\mathscr{F}$ covers Ω, the point x is contained in a cube Q from $\mathscr{F}$. By construc- ›n, Q is not larger than Q_1 and contains the point x of $\bar{Q}_1^c$. Hence, Q is dis- int from Q_1. Therefore, again by construction, Q is not larger than Q_2 and ntains the point x of $\bar{Q}_2^c$, hence is disjoint from Q_2. Continuing in this way, : see that Q is disjoint from $Q_1, Q_2, \ldots, Q_K$. But this is impossible because

then the process would not have terminated after K steps. The cubes $\bar{Q}_k$, $(k = 1, 2, \ldots, K)$, therefore cover Ω. Consequently,

$$|\Omega| \le \sum_{k=1}^{K} |\bar{Q}_k| = 3^n \sum_{k=1}^{K} |Q_k|,$$

and, as we remarked above, this establishes the lemma. ■

Theorem 3.3. *If f belongs to $L^1(\mathbf{R}^n)$, then*

$$t(Mf)^*(t) \le 4^n \|f\|_{L^1}, \qquad (t > 0). \tag{3.4}$$

Proof. Suppose first that f has compact support, in which case it is clear that $(Mf)(x) = 0(|x|^{-n})$ as $|x| \to \infty$. In particular, for each $\lambda > 0$, the set $E_\lambda = \{x \in \mathbf{R}^n : (Mf)(x) > \lambda\}$ has finite measure. For each x in E_λ, the definition (3.1) shows that there is a cube Q_x containing x such that

$$\lambda |Q_x| < \int_{Q_x} |f(y)|\, dy. \tag{3.5}$$

The collection of all such cubes Q_x covers E_λ, so by Lemma 3.2 there exist finitely many disjoint cubes, say $Q_1, Q_2, \ldots, Q_K$, from this collection such that

$$\sum_{k=1}^{K} |Q_k| \ge 4^{-n}|E_\lambda|. \tag{3.6}$$

Hence, combining (3.5) and (3.6), we obtain

$$|E_\lambda| \le 4^n \sum_{k=1}^{K} |Q_k| \le \frac{4^n}{\lambda} \sum_{k=1}^{K} \int_{Q_k} |f(y)|\, dy \le \frac{4^n}{\lambda} \|f\|_{L^1}.$$

Since $|E_\lambda|$ is the distribution function of Mf, this estimate is equivalent to the desired result (3.4) for the decreasing rearrangement $(Mf)^*$ (see Figure 6).

In the general case of an integrable function f, we may select an increasing sequence of nonnegative simple functions $f_k \uparrow |f|$ a.e. It is an easy consequence of the monotone convergence theorem that $Mf_k \uparrow Mf$ a.e., and so it follows from II.(1.17) that $(Mf_k)^* \uparrow (Mf)^*$. Since $\|f_k\|_{L^1} \uparrow \|f\|_{L^1}$, we see that (3.4), which by the argument above holds for each f_k, persists also for f. This completes the proof. ■

If $g = Mf$, the estimate (3.4) shows that the areas of the rectangles lying below the graph of g^* are uniformly bounded (cf. Figure 8). This condition, that $\sup_t tg^*(t)$ be finite, is clearly weaker than integrability, being satisfied by every L^1-function, and by nonintegrable functions such as $1/|x|$. The collection of all such functions is referred to as *weak-L^1*. We shall have much to say about

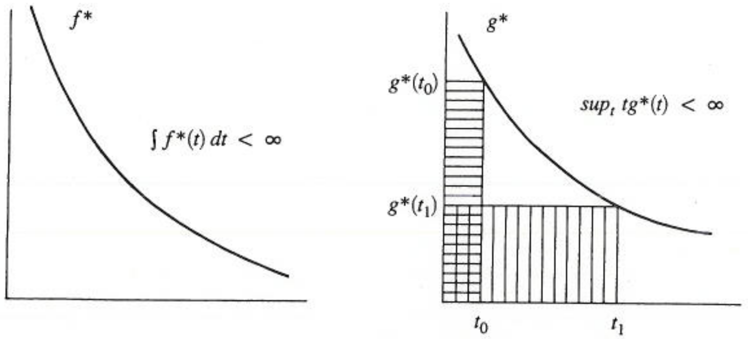

Figure 8. L^1 and weak-L^1.

ch spaces later (cf. Definition V.7.13). For the present, we note simply that e maximal function Mf is in weak-L^1 whenever f belongs to L^1. This weak-pe property of the maximal operator has an important application to the fferentiation of integrals.

ieorem 3.4 (Lebesgue's differentiation theorem). *If f is a locally inte-able function on $\mathbf{R}^n$, then*

$$\lim_{\substack{|Q|\to 0\\ Q\ni x}} \frac{1}{|Q|}\int_Q |f(y)-f(x)|\,dy = 0, \tag{3.7}$$

almost every x in $\mathbf{R}^n$.

oof. Because of the local nature of the result, it is clear that once it is ablished for integrable f, then it will follow also for all locally integrable f. us, we may suppose $f \in L^1(\mathbf{R}^n)$. For each x in $\mathbf{R}^n$, let

$$(\Omega f)(x) = \limsup_{\substack{|Q|\to 0\\ Q\ni x}} \left\{\frac{1}{|Q|}\int_Q |f(y)-f(x)|\,dy\right\}.$$

e theorem will be proved if we show $(\Omega f)(x) = 0$ a.e. in $\mathbf{R}^n$.)bserve that

$$(\Omega f)(x) \le (Mf)(x) + |f(x)|,$$

for any $t > 0$ (cf. II.(1.16)),

$$(\Omega f)^*(t) \le (Mf)^*\left(\frac{t}{2}\right) + f^*\left(\frac{t}{2}\right).$$

Hence, using the weak-type property (3.4) of M and the fact that

$$f^*\left(\frac{t}{2}\right) \le f^{**}\left(\frac{t}{2}\right) \le \left(\frac{2}{t}\right)\|f\|_{L^1},$$

we have

$$(\Omega f)^*(t) \le \frac{c}{t}\|f\|_{L^1}, \tag{3.8}$$

with c depending only on n.

It follows from the fundamental theorem of calculus that $\Omega f = \Omega(f-h)$ whenever h is a continuous function (of compact support, say) so, applying (3.8) to $f-h$ instead of f, we obtain

$$(\Omega f)^*(t) \le \frac{c}{t}\|f-h\|_{L^1}.$$

But the right-hand side may be made as small as we please since the continuous functions of compact support are dense in L^1. Hence $(\Omega f)^*(t) = 0$ for all $t > 0$, and from this we conclude that $\Omega f = 0$ a.e. in $\mathbf{R}^n$. ■

Corollary 3.5. *If f is locally integrable in $\mathbf{R}^n$, then*

$$\lim_{\substack{|Q|\to 0\\ Q\ni x}} \frac{1}{|Q|}\int_Q f(y)\,dy = f(x), \tag{3.9}$$

for almost every x in $\mathbf{R}^n$.

Corollary 3.6. *If f is locally integrable in $\mathbf{R}^n$, then*

$$|f(x)| \le (Mf)(x), \tag{3.10}$$

for almost every x in $\mathbf{R}^n$.

Before proceeding further with our analysis of the maximal function we shall need another covering lemma. In this result, the cover is constructed from the collection of *dyadic cubes*, that is, the cubes formed by means of dilations and contractions by a factor of two of the basic partition of $\mathbf{R}^n$ into unit cubes with vertices at the lattice points.

Lemma 3.7. *Let Ω be an open subset of $\mathbf{R}^n$ with finite measure. Then there is a sequence of dyadic cubes $Q_1, Q_2, \ldots$, with pairwise disjoint interiors, that covers*

Ω and satisfies

(i) $Q_k \cap \Omega^c \neq \varnothing, \quad (k = 1, 2, \ldots);$

(ii) $|\Omega| \leq \sum_{k=1}^{\infty} |Q_k| \leq 2^n|\Omega|.$

Proof. Since Ω is open, there exists, for each x in Ω, a dyadic cube, say $Q(x)$, of smallest diameter, which contains x and has nonempty intersection with Ω^c. Subdivide $Q(x)$ into 2^n congruent dyadic subcubes and select any one, say $\tilde{Q}(x)$, hat contains x. Then $\tilde{Q}(x) \subset \Omega$ and so

$$2^{-n}|Q(x)| = |\tilde{Q}(x)| = |\tilde{Q}(x) \cap \Omega| \leq |Q(x) \cap \Omega|. \tag{3.11}$$

Now, pairs of dyadic cubes have the property that if their interiors have non-mpty intersection, then one cube is contained in the other. Hence, since Ω as finite measure, every point x in Ω is contained in a maximal cube from he collection $\{Q(y): y \in \Omega\}$. There are at most countably many such maximal ubes (since there are only countably many dyadic cubes), which we list as $Q_1, Q_2, \ldots$, and they have pairwise disjoint interiors. They evidently cover Ω nd so the first inequality in (ii) holds. Property (i) holds by construction. urthermore, every Q_k satisfies an inequality of the type (3.11), and so

$$\sum_k |Q_k| \leq 2^n \sum_k |Q_k \cap \Omega| = 2^n|\Omega|.$$

his establishes the second of the inequalities in (ii) and hence completes the roof. ■

The function $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds$ is the average of f^* over the interval $(0, t)$. This is maximal among all averages of f^* over intervals containing t ecause f^* is decreasing. Hence, f^{**} is the Hardy-Littlewood maximal unction of f^*. The next result demonstrates the equivalence of the maximal unction of the decreasing rearrangement (namely, f^{**}) and the decreasing earrangement of the maximal function (namely, $(Mf)^*$).

heorem 3.8 *There are constants c and c', depending only on n, such that*

$$c(Mf)^*(t) \leq f^{**}(t) \leq c'(Mf)^*(t), \qquad (t > 0), \tag{3.12}$$

r every locally integrable function f on $\mathbf{R}^n$.

oof. Fix $t > 0$. For the left-hand inequality, we may suppose $f^{**}(t) < \infty$, herwise there is nothing to prove. In that case, by Theorem II.6.2, given > 0, there are functions g_t in L^1 and h_t in L^∞ such that $f = g_t + h_t$ and

$$\|g_t\|_{L^1} + t\|h_t\|_{L^\infty} \leq tf^{**}(t) + \varepsilon. \tag{3.13}$$

Then, by (3.4) and (3.2), for any $s > 0$,

$$(Mf)^*(s) \leq (Mg_t)^*\left(\frac{s}{2}\right) + (Mh_t)^*\left(\frac{s}{2}\right) \leq \frac{c}{s}\|g_t\|_{L^1} + \|h_t\|_{L^\infty}$$

$$\leq \frac{c}{s}(\|g_t\|_{L^1} + s\|h_t\|_{L^\infty}).$$

Putting $s = t$, using (3.13), and letting $\varepsilon \to 0$, we obtain the first of the inequalities in (3.12).

For the right-hand inequality in (3.12), we may suppose $(Mf)^*(t) < \infty$, otherwise there is nothing to prove. The lower semi-continuity of Mf ensures that the set

$$\Omega = \{x \in \mathbf{R}^n : (Mf)(x) > (Mf)^*(t)\}$$

is open, and we have $|\Omega| \leq t$ because Mf and $(Mf)^*$ are equimeasurable. Applying Lemma 3.7, we obtain a sequence of cubes $Q_1, Q_2, \ldots$, with pairwise disjoint interiors, that cover Ω and satisfy

$$Q_k \cap \Omega^c \neq \varnothing, \qquad (k = 1, 2 \ldots) \tag{3.14}$$

and

$$\sum |Q_k| \leq 2^n|\Omega| \leq 2^n t. \tag{3.15}$$

With $F = (\bigcup_k Q_k)^c$, we set

$$g = \sum_k f\chi_{Q_k}, \qquad h = f\chi_F,$$

so $f = g + h$. Then the subadditivity of $f \to f^{**}$ (Theorem II.3.4) gives

$$f^{**}(t) \leq g^{**}(t) + h^{**}(t) \leq \frac{1}{t}\|g\|_{L^1} + \|h\|_{L^\infty}. \tag{3.16}$$

Now, by (3.14), each cube Q_k contains a point of Ω^c, and at such a point the maximal function has value at most $(Mf)^*(t)$ because of the way in which Ω is defined. Thus,

$$\frac{1}{|Q_k|}\int_{Q_k} |f(y)|\,dy \leq (Mf)^*(t), \qquad (k = 1, 2, \ldots).$$

The L^1-norm of g may therefore be estimated by

$$\|g\|_{L^1} = \sum_k \int_{Q_k} |f(y)|\,dy \leq \sum_k |Q_k|(Mf)^*(t).$$

Hence, using (3.15), we have

$$\|g\|_{L^1} \leq 2^n t(Mf)^*(t). \tag{3.17}$$

n the other hand, the set F is contained in Ω^c and so the maximal function is ounded by $(Mf)^*(t)$ on F. Hence, using (3.10), we have

$$\|h\|_{L^\infty} = \|f\chi_F\|_{L^\infty} \le \|(Mf)\chi_F\|_{L^\infty} \le (Mf)^*(t).$$

ombining the last estimate with (3.16) and (3.17), we therefore obtain the ght-hand inequality in (3.12). ■

The last result simplifies the problem of establishing the boundedness the maximal operator on rearrangement-invariant spaces. Indeed, it re- ıces the problem to that of establishing boundedness of f^{**}, which often is sier since f^{**} is a simple average rather than a supremum of averages. e shall elaborate on this further in Section 5 in the context of arbitrary arrangement-invariant spaces. For the present, however, we shall content ırselves with establishing the boundedness of the maximal operator in the -setting. The following inequalities, known simply as *Hardy's inequalities,* ll be crucial.

mma 3.9 (G. H. Hardy). *Let ψ be a nonnegative measurable function on ∞) and suppose $-\infty < \lambda < 1$ and $1 \le q \le \infty$. Then*

$$\left\{\int_0^\infty \left(t^\lambda \frac{1}{t}\int_0^t \psi(s)\,ds\right)^q \frac{dt}{t}\right\}^{1/q} \le \frac{1}{1-\lambda}\left\{\int_0^\infty (t^\lambda \psi(t))^q \frac{dt}{t}\right\}^{1/q} \tag{3.18}$$

d

$$\left\{\int_0^\infty \left(t^{1-\lambda} \int_t^\infty \psi(s)\frac{ds}{s}\right)^q \frac{dt}{t}\right\}^{1/q} \le \frac{1}{1-\lambda}\left\{\int_0^\infty (t^{1-\lambda}\psi(t))^q \frac{dt}{t}\right\}^{1/q} \tag{3.19}$$

th the obvious modification if $q = \infty$).

oof. Writing $\psi(s) = s^{-\lambda/q'}s^{\lambda/q'}\psi(s)$ and applying Hölder's inequality, we tain

$$\frac{1}{t}\int_0^t \psi(s)\,ds \le \left(\frac{1}{t}\int_0^t s^{-\lambda}\,ds\right)^{1/q'}\left(\frac{1}{t}\int_0^t s^{\lambda q/q'}\psi(s)^q\,ds\right)^{1/q}$$

$$= (1-\lambda)^{-1/q'} t^{-\lambda/q' - 1/q}\left(\int_0^t s^{\lambda(q-1)}\psi(s)^q\,ds\right)^{1/q}.$$

nce, by an interchange in the order of integration,

$$\int_0^\infty \left(t^\lambda \frac{1}{t}\int_0^t \psi(s)\,ds\right)^q \frac{dt}{t} \le (1-\lambda)^{1-q}\int_0^\infty t^{\lambda-2}\int_0^t s^{\lambda(q-1)}\psi(s)^q\,ds\,dt$$

$$= (1-\lambda)^{1-q}\int_0^\infty s^{\lambda(q-1)}\psi(s)^q\int_s^\infty t^{\lambda-2}\,dt\,ds.$$

Performing the integration over t and taking q-th roots, we obtain (3.18). The proof of (3.19) is similar and is left as an exercise. ■

Now we can establish the L^p-boundedness of the Hardy-Littlewood maximal operator.

Theorem 3.10 (Hardy-Littlewood maximal theorem). *Let $1 < p \le \infty$ and suppose f belongs to $L^p(\mathbf{R}^n)$. Then Mf belongs to $L^p(\mathbf{R}^n)$ and*

$$\|Mf\|_{L^p} \le c\|f\|_{L^p}, \tag{3.20}$$

where c is a constant depending only on p and n.

Proof. Since $p > 1$, we may apply the first part of Lemma 3.9 with $\lambda = 1/p$ and $q = p$. In that case, we obtain from (3.12) and (3.18), with c depending only on n,

$$\begin{aligned}\|Mf\|_{L^p(\mathbf{R}^n)} &= \left(\int_0^\infty (Mf)^*(t)^p\,dt\right)^{1/p}\\ &\le c\left(\int_0^\infty\left(\frac{1}{t}\int_0^t f^*(s)\,ds\right)^p dt\right)^{1/p}\\ &\le cp'\left(\int_0^\infty f^*(t)^p\,dt\right)^{1/p}\\ &= cp'\|f\|_{L^p(\mathbf{R}^n)}.\end{aligned}$$ ■

The last result may be regarded as a rudimentary interpolation theorem for the maximal operator. The crucial ingredient in the proof (Theorem 3.8) asserts that

$$(Mf)^*(t) \le cf^{**}(t), \qquad (t > 0), \tag{3.21}$$

for every locally integrable function f. Encoded in this inequality are the properties that M is bounded on L^∞:

$$\|Mf\|_{L^\infty} = \sup_{t>0}(Mf)^*(t) \le c\cdot\sup_{t>0} f^{**}(t) = c\|f\|_{L^\infty},$$

and that M is bounded from L^1 into weak-L^1:

$$\begin{aligned}\|Mf\|_{\text{weak-}L^1} &= \sup_{t>0} t(Mf)^*(t) \le \sup_{t>0} ctf^{**}(t)\\ &= \sup_{t>0} c\int_0^t f^*(s)\,ds = c\|f\|_{L^1}.\end{aligned}$$

These are the hypotheses of the interpolation theorem. The conclusion is the L^p-boundedness of the maximal operator for $1 < p < \infty$ (cf. Theorem 3.10). Note that the proof uses only (3.21) and the appropriate Hardy inequality.

We shall show in the next section that an inequality analogous to (3.21) holds for the Hilbert transform, which can then be interpolated in the same elementary fashion. Ultimately, we shall identify a large class of operators for which such interpolation is possible, the end result being the celebrated Marcinkiewicz interpolation theorem.

4. THE HILBERT TRANSFORM

The Hilbert transform is a linear operator which arises from the study of boundary values of the real and imaginary parts of analytic functions. Questions involving the Hilbert transform arise therefore from the utilization of complex methods in Fourier analysis, for example. In particular, the Hilbert transform plays the decisive role in questions of norm-convergence of Fourier series and Fourier integrals. These applications will be deferred for the time being, as will any discussion of complex methods. Instead, a direct real variable approach will be used. This will establish the existence and L^p-boundedness of the Hilbert transform with a minimum of effort and will also serve to motivate the weak-type interpolation theory to be developed in the next section.

Definition 4.1. Let f be a locally integrable function on the real line $\mathbf{R}$. The *Hilbert transform* Hf of f is defined by the principal-value integral

$$(Hf)(x) = \text{p.v.}\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{dt}{x-t} \tag{4.1}$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-t|\geq\varepsilon} f(t)\frac{dt}{x-t}, \qquad (x\in\mathbf{R}),$$

provided the limit exists a.e. The operator $H: f \to Hf$ is also referred to as the *Hilbert transform.*

Even for relatively "nice" functions, such as the continuous functions of compact support, say, it is by no means obvious that the principal-value integral in (4.1) exists a.e. Thus, establishing existence will be one of our first priorities. As a starting point, notice that if f is the characteristic function of a finite interval (a, b), then a direct integration shows that

$$H\chi_{(a,b)}(x) = \frac{1}{\pi}\log\left|\frac{x-a}{x-b}\right|, \qquad (x \neq a, b). \tag{4.2}$$

It follows that Hf exists a.e. whenever f is a step function (that is, a function which assumes only finitely many values, each being assumed on a finite disjoint union of intervals of finite length).

There is an analogy between the problems of the existence of the Hilbert transform and the existence of the limit in the Lebesgue differentiation theorem (Theorem 3.4). In the latter situation, the a.e. existence of the limit (of certain integral averages) was known for a dense subset of L^1 (the continuous functions of compact support) and the result was extended to all of L^1 by establishing control (Theorem 3.3) over the corresponding maximal operator (the Hardy-Littlewood maximal operator M). For the Hilbert transform, the dense subset of L^1 consists of the step functions, and in order to extend to all of L^1 the a.e. existence of the limit of

$$(H_\varepsilon f)(x) = \frac{1}{\pi}\int_{|x-t|\geq\varepsilon} f(t)\frac{dt}{x-t}, \qquad (x\in\mathbf{R}), \tag{4.3}$$

as $\varepsilon \to 0$, it will be necessary to consider the following maximal operator.

Definition 4.2. Let f be a locally integrable function on $\mathbf{R}$. The *maximal Hilbert transform* $\mathscr{H}f$ of f is defined by

$$(\mathscr{H}f)(x) = \sup_{\varepsilon>0}|(H_\varepsilon f)(x)|, \qquad (x\in\mathbf{R}). \tag{4.4}$$

The operator $\mathscr{H}: f \to \mathscr{H}f$ is called the *maximal Hilbert transform.*

The necessary estimates for the maximal Hilbert transform will be derived from the following pair of elementary lemmas, which provide information on the distribution functions of certain rational functions.

Lemma 4.3. *Suppose a_i, b_i, $(i = 1, 2, \ldots, n)$, are real numbers satisfying $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$ and let g be the rational function*

$$g(x) = \prod_{k=1}^{n}\frac{x-a_k}{x-b_k}, \qquad (x\in\mathbf{R}). \tag{4.5}$$

If $\Delta \neq 1$, then the equation $g(x) = \Delta$ has n distinct roots $r_1, r_2, \ldots, r_n$, which satisfy

$$\sum_{k=1}^{n} b_k = \sum_{k=1}^{n} r_k + (1-\Delta)^{-1}\sum_{k=1}^{n}(b_k - a_k). \tag{4.6}$$

Furthermore, if $\Delta > 1$, then

$$(\Delta - 1)|\{g > \Delta\}| = (\Delta + 1)|\{g < -\Delta\}| = \sum_{k=1}^{n}(b_k - a_k). \tag{4.7}$$

proof of Theorem 3.6. The full power of Lemma 3.1 would not be needed in this particular application—as in the proof above, the weaker three-lines theorem would suffice.

4. THE MARCINKIEWICZ INTERPOLATION THEOREM

Consider the averaging operator A defined on $L^1(0,1)$ by

$$(Af)(t) = \frac{1}{t}\int_0^t f(s)\,ds, \qquad (0 < t < 1). \tag{4.1}$$

It follows at once from Hardy's inequality (Lemma III.3.9) that A is a bounded linear operator on L^p (into itself) for $1 < p < \infty$. If we wished to establish this result by appealing to the Riesz-Thorin interpolation theorem, we would first need to verify that A is bounded on L^∞ and on L^1. The L^∞-boundedness follows immediately from (4.1), but A is not bounded on L^1, as may easily be seen by considering a decreasing function of the form $f(s) = s^{-1}(\log s)^{-2}$ near the origin and observing that Tf fails to be integrable there. Thus, even in this simple case, the Riesz-Thorin theorem does not apply.

The desired interpolation can still be accomplished, but by a quite different technique introduced by J. Marcinkiewicz in 1939. The Marcinkiewicz interpolation theorem is best formulated in the larger context of a two-parameter family of spaces $L^{p,q}$ (the Lorentz spaces), which generalize the Lebesgue spaces L^p. Our first task therefore will be to define the Lorentz spaces and derive some of their elementary properties.

Definition 4.1. Let (R,μ) be a totally σ-finite measure space and suppose $0 < p,\ q \le \infty$. The *Lorentz space* $L^{p,q} = L^{p,q}(R,\mu)$ consists of all f in $\mathcal{M}_0(R,\mu)$ for which the quantity

$$\|f\|_{p,q} = \begin{cases} \left\{\int_0^\infty [t^{1/p}f^*(t)]^q \dfrac{dt}{t}\right\}^{1/q}, & (0 < q < \infty), \\ \sup_{0<t<\infty} \{t^{1/p}f^*(t)\}, & (q = \infty), \end{cases} \tag{4.2}$$

finite.

It is clear from Proposition II.1.8 that the Lorentz space $L^{p,p}$, $(0 < p \le \infty)$, coincides with the Lebesgue space L^p, and

$$\|f\|_{p,p} = \|f\|_p, \qquad (f \in L^p). \tag{4.3}$$

Note also that the space $L^{\infty,q}$, for finite q, is trivial, in the sense that it contains only the zero-function.

The next result shows that, for any fixed p, the Lorentz spaces $L^{p,q}$ increase as the secondary exponent q increases.

Proposition 4.2. *Suppose $0 < p \le \infty$ and $0 < q \le r \le \infty$. Then*

$$\|f\|_{p,r} \le c\|f\|_{p,q}, \tag{4.4}$$

for all f in $\mathcal{M}_0(R,\mu)$, where c is a constant depending only on p, q, and r. In particular, there is the embedding $L^{p,q} \hookrightarrow L^{p,r}$.

Proof. We may assume $p < \infty$ and $q < r$ since in the other cases there is nothing to prove. Using the fact that f^* is decreasing, we have

$$\begin{aligned} t^{1/p}f^*(t) &= \left\{\frac{p}{q}\int_0^t [s^{1/p}f^*(t)]^q \frac{ds}{s}\right\}^{1/q} \\ &\le \left\{\frac{p}{q}\int_0^t [s^{1/p}f^*(s)]^q \frac{ds}{s}\right\}^{1/q} \\ &\le \left(\frac{p}{q}\right)^{1/q} \|f\|_{p,q}. \end{aligned}$$

Hence, taking the supremum over all $t > 0$, we obtain

$$\|f\|_{p,\infty} \le \left(\frac{p}{q}\right)^{1/q} \|f\|_{p,q}. \tag{4.5}$$

This establishes (4.4) in the case $r = \infty$.

In the remaining case where $r < \infty$, we have

$$\begin{aligned} \|f\|_{p,r} &= \left\{\int_0^\infty [t^{1/p}f^*(t)]^{r-q+q} \frac{dt}{t}\right\}^{1/r} \\ &\le \|f\|_{p,\infty}^{1-q/r}\|f\|_{p,q}^{q/r}. \end{aligned}$$

When combined with (4.5), this produces (4.4) with the constant $c = (p/q)^{(r-q)/rq}$. ∎

Inclusion relations among $L^{p,q}$-spaces, with p varying, are like those for the Lebesgue spaces L^p, in that they depend on the structure of the underlying measure space. The secondary exponent q is not involved. Thus, if $0 < p < r \le \infty$ and $0 < q, s \le \infty$, then $L^{r,s} \hookrightarrow L^{p,q}$ on finite measure spaces and $\ell^{p,q} \hookrightarrow \ell^{r,s}$ on discrete measure spaces (the details are left as an exercise).

We next wish to determine for which values of p and q the Lorentz space $L^{p,q}$ may be regarded as a rearrangement-invariant Banach function space. The functional $f \to \|f\|_{p,q}$ is not always a norm, even when $p, q \geq 1$. How-ever, there is the following result.

Theorem 4.3. *Suppose $1 \leq q \leq p < \infty$ or $p = q = \infty$. Then $(L^{p,q}, \|\cdot\|_{p,q})$ is a rearrangement-invariant Banach function space, with upper and lower Boyd ndices both equal to $1/p$.*

Proof. The result is clear when $p = q = 1$ or $p = q = \infty$ since then $L^{p,q}$ educes to the Lebesgue spaces L^1 and L^∞, respectively. Hence, we may as-ume that $1 < p < \infty$ and $1 \leq q \leq p$. In that case, with $q' = q/(q-1)$, we iave from (4.2) and II.(4.6),

$$\|f+g\|_{p,q} = \left\{\int_0^\infty [t^{1/p-1/q}(f+g)^*(t)]^q\,dt\right\}^{1/q}$$

$$= \sup\left\{\int_0^\infty t^{1/p-1/q}(f+g)^*(t)h^*(t)\,dt : \|h\|_{L^{q'}(0,\infty)} = 1\right\}. \tag{4.6}$$

he hypothesis $q \leq p$ implies that $t^{1/p-1/q}$, hence also $t^{1/p-1/q}h^*(t)$, is ecreasing. Hence, since $(f+g)^* \prec f^* + g^*$ (Theorem II.3.4), we may apply ardy's lemma (Proposition II.3.6) and then Hölder's inequality to obtain

$$\begin{aligned}\int_0^\infty t^{1/p-1/q}(f+g)^*(t)h^*(t)\,dt &\leq \int_0^\infty t^{1/p-1/q}f^*(t)h^*(t)\,dt \\ &\quad + \int_0^\infty t^{1/p-1/q}g^*(t)h^*(t)\,dt \\ &\leq \left\{\int_0^\infty t^{q/p-1}f^*(t)^q\,dt\right\}^{1/q}\|h\|_{q'} \\ &\quad + \left\{\int_0^\infty t^{q/p-1}g^*(t)^q\,dt\right\}^{1/q}\|h\|_{q'} \\ &= \|f\|_{p,q} + \|g\|_{p,q},\end{aligned}$$

nce h has norm 1 in $L^{q'}(0,\infty)$. This, together with (4.6), establishes the triangle equality for $\|\cdot\|_{p,q}$. The remaining properties of a Banach function norm (cf. efinition I.1.1) are easy to verify for $\|\cdot\|_{p,q}$, and the rearrangement-invariance obvious.

To compute the Boyd indices, let $t > 0$ and consider the dilation operator

$(E_t g)(s) = g(st)$ (cf. Definition III.5.10). If $q < \infty$, a change of variables in (4.2) gives

$$\begin{aligned}\|E_t(f^*)\|_{p,q} &= \left\{\int_0^\infty [s^{1/p}f^*(st)]^q \frac{ds}{s}\right\}^{1/q} \\ &= \left\{\int_0^\infty [t^{-1/p}u^{1/p}f^*(u)]^q \frac{du}{u}\right\}^{1/q} \\ &= t^{-1/p}\|f\|_{p,q}.\end{aligned}$$

The operator norm $h(t)$ of $E_{1/t}$ is therefore equal to $t^{1/p}$. It now follows at once from Definition III.5.12 that both Boyd indices are equal to $1/p$. The proof is similar for $q = \infty$. ∎

Although the restriction $q \leq p$ in the previous result is necessary, it can be circumvented in the case $p > 1$ by replacing $\|\cdot\|_{p,q}$ with an equivalent functional which is a norm for all $q \geq 1$. The trick is simply to replace f^* with f^{**} in the definition (4.2) of $\|f\|_{p,q}$.

Definition 4.4. Suppose $1 < p \leq \infty$ and $0 < q \leq \infty$. If $f \in \mathcal{M}_0(R,\mu)$, let

$$\|f\|_{(p,q)} = \begin{cases}\left\{\int_0^\infty [t^{1/p}f^{**}(t)]^q \frac{dt}{t}\right\}^{1/q} & (0 < q < \infty) \\ \sup_{0<t<\infty} \{t^{1/p}f^{**}(t)\} & (q = \infty).\end{cases} \tag{4.7}$$

Lemma 4.5. *If $1 < p \leq \infty$ and $1 \leq q \leq \infty$, then*

$$\|f\|_{p,q} \leq \|f\|_{(p,q)} \leq p'\|f\|_{p,q} \tag{4.8}$$

for all $f \in \mathcal{M}_0(R,\mu)$, where $p' = p/(p-1)$. In particular, $L^{p,q}$ consists of all f for which $\|f\|_{(p,q)}$ is finite.

Proof. The first inequality in (4.8) is an immediate consequence of the definitions (4.2) and (4.7), and the fact that $f^* \leq f^{**}$. The second follows directly from Hardy's inequality III.(3.18). ∎

Since $f \to f^{**}$ is subadditive (Theorem II.3.4), the triangle inequality for $\|\cdot\|_{(p,q)}$ follows immediately from Minkowski's inequality. Hence, we have the following result.

Theorem 4.6. *If $1 < p < \infty$, $1 \leq q \leq \infty$ or if $p = q = \infty$, then $(L^{p,q}, \|\cdot\|_{(p,q)})$*

s a rearrangement-invariant Banach function space with lower and upper Boyd ndices both equal to $1/p$.

It is worth noting that if $1 \le p < \infty$, the space $L^{p,1}$, equipped with the ıorm $\|\cdot\|_{p,1}$, coincides with the Lorentz space $\Lambda(L^p)$, and (cf. II.(5.18))

$$\|f\|_{p,1} = \int_0^\infty t^{1/p} f^*(t) \frac{dt}{t} = p\|f\|_{\Lambda(L^p)}. \tag{4.9}$$

On the other hand, if $1 < p \le \infty$, the space $L^{p,\infty}$, equipped with the modiied norm $\|\cdot\|_{(p,\infty)}$, coincides with the Lorentz space $M(L^p)$, and (cf. II.(5.17))

$$\|f\|_{(p,\infty)} = \sup_{t>0} t^{1/p} f^{**}(t) = \|f\|_{M(L^p)}. \tag{4.10}$$

In particular, if $1 < p < \infty$, then Theorem II.5.13 shows that $L^{p,1}$ and $L^{p,\infty}$, hen suitably normed, are respectively the smallest and the largest of all earrangement-invariant spaces having the same fundamental function as L^p. hese spaces will play a particularly important role in the weak-type ıterpolation theory to be discussed later. First, however, let us determine the ssociate and dual spaces of the Lorentz spaces $L^{p,q}$.

heorem 4.7. *Let (R, μ) be a resonant measure space and suppose $1 < p < \infty$, $\le q \le \infty$ (or $p = q = 1$ or $p = q = \infty$). Then the associate space of $L^{p,q}(R, \mu)$;, up to equivalence of norms, the Lorentz space $L^{p',q'}(R, \mu)$, where $1/p + 1/p' =$ $/q + 1/q' = 1$.*

roof. If $p = q = 1$ or if $p = q = \infty$, the result merely reiterates the wellnown fact that the Lebesgue spaces L^1 and L^∞ are mutually associate. Ience, we may assume that $1 < p < \infty$ and $1 \le q \le \infty$. By Hölder's inequaly, we have

$$\left|\int_R fg\,d\mu\right| \le \int_0^\infty f^*(t)g^*(t)\,dt \le \int_0^\infty f^{**}(t)g^{**}(t)\,dt$$

$$= \int_0^\infty [t^{1/p} f^{**}(t)][t^{1/p'} g^{**}(t)] \frac{dt}{t}$$

$$\le \|f\|_{(p,q)}\|g\|_{(p',q')}.$$

Ience, taking the supremum over all f of norm at most 1, we obtain

$$\|g\|_{(L^{p,q})'} \le \|g\|_{(p',q')}, \qquad (g \in L^{p',q'}). \tag{4.11}$$

To establish an inequality of this type in the opposite direction, it suffices y the Luxemburg representation theorem (Theorem II.4.10) to do so for the

measure space $(\mathbf{R}^+, m)$ and functions g on $\mathbf{R}^+$ for which $g = g^*$. Suppose first that $1 < q < \infty$ and that $g = g^*$ is a simple function. Let

$$f(t) = \int_{t/2}^\infty \phi(s) \frac{ds}{s}, \qquad (t > 0),$$

where

$$\phi(s) = s^{q'/p' - 1} g^*(s)^{q'-1}, \qquad (s > 0).$$

Then $f = f^*$ and, by (4.2),

$$\begin{aligned} \|g\|_{p',q'}^{q'} &= \int_0^\infty \phi(s) g^*(s)\,ds \\ &\le c\int_0^\infty \left\{\int_{s/2}^s \phi(t)\frac{dt}{t}\right\} g^*(s)\,ds \\ &\le c\int_0^\infty f^*(s)g^*(s)\,ds \\ &\le c\|f\|_{L(p,q)}\|g\|_{(L^{p,q})'}, \end{aligned} \tag{4.12}$$

where c is a constant depending only on p and q. However, using Lemma 4.5 and Hardy's inequality (Lemma III.3.9), we have

$$\begin{aligned} \|f\|_{(p,q)} \le c\|f\|_{p,q} &= c\left\{\int_0^\infty \left[t^{1/p} \int_{t/2}^\infty \phi(s)\frac{ds}{s}\right]^q \frac{dt}{t}\right\}^{1/q} \\ &\le c\left\{\int_0^\infty [t^{1/p}\phi(t)]^q \frac{dt}{t}\right\}^{1/q} \\ &= c\left\{\int_0^\infty [t^{1/p'} g^*(t)]^{q'} \frac{dt}{t}\right\}^{1/q} = c\|g\|_{p',q'}^{q'/q}. \end{aligned}$$

Combining this with (4.12), we obtain

$$\|g\|_{p',q'} \le \|g\|_{(L^{p,q})'}, \tag{4.13}$$

with c depending only on p and q. This holds for all simple $g\,(=g^*)$ and hence, by the Fatou property and rearrangement-invariance, for all g in $(L^{p,q})'$. Hence, (4.11) and (4.13) together complete the proof for the case $1 < q < \infty$. The proof for $q = 1$ and $q = \infty$ is similar. ■

Corollary 4.8. *If (R, μ) is resonant and if $1 < p < \infty$, $1 \le q < \infty$ (or $p = q = 1$), then the dual space $(L^{p,q})^*$ of $L^{p,q}(R, \mu)$ can be identified with the associate space $L^{p',q'}$ (with equivalent norms).*

roof. For $p = q = 1$, the result merely asserts that L^∞ is identifiable as he dual of L^1. In the remaining case where $1 < p < \infty$, it is easy to see di-ectly from (4.7) that $L^{p,q}$ has absolutely continuous norm when $1 \le q < \infty$. Ience, the desired result follows at once from the preceding theorem and Corollary I.4.3. ■

With these preliminaries disposed of, let us now return to the main topic f weak-type interpolation. Recall from Definition III.5.1 that if $1 \le p_0 <$ $_1 \le \infty$, $1 \le q_0, q_1 \le \infty$ with $q_0 \ne q_1$, and if m denotes the slope

$$m = \frac{\dfrac{1}{q_0} - \dfrac{1}{q_1}}{\dfrac{1}{p_0} - \dfrac{1}{p_1}} \tag{4.14}$$

f the interpolation segment $\sigma = [(1/p_0, 1/q_0), (1/p_1, 1/q_1)]$, then the Calderón perator S_σ is defined by

$$(S_\sigma f)(t) = t^{-1/q_0} \int_0^{t^m} s^{1/p_0} f(s) \frac{ds}{s} + t^{-1/q_1} \int_{t^m}^{\infty} s^{1/p_1} f(s) \frac{ds}{s} \tag{4.15}$$

r each $t > 0$. Furthermore (cf. Definition III.5.4), an operator T is of joint eak type $(p_0, q_0; p_1, q_1)$ if

$$(Tf)^*(t) \le M S_\sigma(f^*)(t), \qquad (t > 0), \tag{4.16}$$

r all f for which the right-hand side is finite, the least value of M being the eak-type $(p_0, q_0; p_1, q_1)$ norm of T.

The Lorentz space structure will enable us to separate this joint weak-type roperty into two individual weak-type conditions for the pairs (p_0, q_0) and $_1, q_1)$ (at least when p_0 and p_1 are finite).

efinition 4.9. Let (R, μ) and (S, ν) be totally σ-finite measure spaces and ppose $1 \le p < \infty$ and $1 \le q \le \infty$. Let T be an operator defined on $^{p,1}(R, \mu)$ and taking values in $\mathscr{M}_0(S, \nu)$. Then T is said to be of *weak type* (p, q) it is a bounded operator from $L^{p,1}(R, \mu)$ into $L^{q,\infty}(S, \nu)$, that is, if there is a nstant M such that

$$\|Tf\|_{q,\infty} \le M \|f\|_{p,1} \tag{4.17}$$

r all f in $L^{p,1}$. The least constant M for which (4.17) holds is called the *eak-type* (p, q) *norm* of T.

Explicitly, in terms of the decreasing rearrangement, the estimate (4.17)

asserts that

$$(Tf)^*(t) \le M t^{-1/q} \|f\|_{p,1}, \qquad (t > 0),$$

or, in terms of the distribution function, that

$$\nu\{|Tf| > \lambda\} \le [M \lambda^{-1} \|f\|_{p,1}]^q, \qquad (\lambda > 0).$$

If p is finite, then a linear operator T of strong type (p, q), defined initially on the simple functions, has a unique linear extension to a bounded operator from L^p to L^q. Since Proposition 4.2 shows that $L^{p,1} \hookrightarrow L^p$ and $L^q \hookrightarrow L^{q,\infty}$, it follows that T is also of weak type (p, q).

Lemma 4.10. *Suppose $1 \le p_0 < p_1 < \infty$ and $1 \le q_0, q_1 \le \infty$ with $q_0 \ne q_1$. If S_σ is defined as in* (4.15), *then*

$$t^{1/q_i} S_\sigma(f^*)(t) \le \int_0^\infty s^{1/p_i} f^*(s) \frac{ds}{s}, \qquad (t > 0), \tag{4.18}$$

for all f in $L^{p_i,1}(\mathbf{R}^+)$, $(i = 0, 1)$. In particular, S_σ is of weak types (p_0, q_0) and (p_1, q_1).

Proof. From (4.15), we have

$$t^{1/q_0} S_\sigma(f^*)(t) = \int_0^{t^m} s^{1/p_0} f^*(s) \frac{ds}{s} + t^{1/q_0 - 1/q_1} \int_{t^m}^\infty s^{1/p_1} f^*(s) \frac{ds}{s}, \tag{4.19}$$

where m is given by (4.14). The hypothesis $p_0 < p_1$ implies that

$$t^{1/q_0 - 1/q_1} = (t^m)^{1/p_0 - 1/p_1} \le s^{1/p_0 - 1/p_1}$$

whenever $s \ge t^m$. Hence, applying this estimate to the second term on the right of (4.19), then combining the two integrals, we obtain (4.18) in the case $i = 0$. The proof for $i = 1$ is similar. Since III.(5.7) shows that $[S_\sigma(f)]^* \le S_\sigma(f^*)$, the weak-type assertions follow at once from (4.18). ■

Theorem 4.11. *Suppose $1 \le p_0 < p_1 < \infty$ and $1 \le q_0, q_1 \le \infty$ with $q_0 \ne q_1$. Then a quasilinear operator T is of joint weak type $(p_0, q_0; p_1, q_1)$ if and only if it is of weak types (p_0, q_0) and (p_1, q_1).*

Proof. Suppose T is of weak types (p_0, q_0) and (p_1, q_1). Let $f \in (L^{p_0,1} + L^{p_1,1})(R, \mu)$ and fix $t > 0$. With m given by (4.14), define f_0 and f_1 on R by

$$\begin{aligned} f_1(x) &= \min[|f(x)|, f^*(t^m)] \cdot \operatorname{sgn} f(x) \\ f_0(x) &= f(x) - f_1(x) = [|f(x)| - f^*(t^m)]^+ \cdot \operatorname{sgn} f(x). \end{aligned} \tag{4.20}$$

Then

$$f_1^*(s) = \min[f^*(s), f^*(t^m)], \qquad (s > 0)$$

and so, by (4.2),

$$\|f_1\|_{p_1,1} = p_1 t^{m/p_1} f^*(t^m) + \int_{t^m}^{\infty} s^{1/p_1} f^*(s) \frac{ds}{s}. \tag{4.21}$$

Similarly,

$$f_0^*(s) = [f^*(s) - f^*(t^m)]^+, \qquad (s > 0)$$

and

$$\|f_0\|_{p_0,1} = \int_0^{t^m} s^{1/p_0} f^*(s) \frac{ds}{s} - p_0 t^{m/p_0} f^*(t^m). \tag{4.22}$$

Now suppose that T is quasilinear with constant k (cf. III.(5.8)). Since $f = f_0 + f_1$, we have from Proposition II.1.7,

$$(Tf)^*(t) \le [k(|Tf_0| + |Tf_1|)]^*(t)$$
$$\le k\left[(Tf_0)^*\left(\frac{t}{2}\right) + (Tf_1)^*\left(\frac{t}{2}\right)\right].$$

Furthermore, by (4.17), the weak-type hypotheses on T give

$$(Tf_i)^*\left(\frac{t}{2}\right) \le \left(\frac{t}{2}\right)^{-1/q_i} M_i \|f_i\|_{p_i,1}, \qquad (i = 0, 1). \tag{4.23}$$

Combining these estimates, we obtain

$$(Tf)^*(t) \le c\left\{\frac{t^{-1/q_0}}{p_0}\|f_0\|_{p_0,1} + \frac{t^{-1/q_1}}{p_1}\|f_1\|_{p_1,1}\right\} \tag{4.24}$$

with $c = k \cdot \max_i p_i M_i 2^{1/q_i}$. Incorporating (4.21), (4.22) into (4.24) and observing that the terms in $f^*(t^m)$ cancel, we find that the right-hand side of (4.24) is majorized by $cS_\sigma(f^*)(t)$. Hence, T is of joint weak type $(p_0, q_0; p_1, q_1)$.

The converse is an immediate consequence of (4.16) and Lemma 4.10. ■

***Remarks* 4.12.** (a) If T is of weak type $(p_0, q_0; \infty, q_1)$, with $p_0 < \infty$, we can conclude from the argument above only that T is of weak type (p_0, q_0).
(b) A simple modification of the proof above shows that if T is of weak type (p_0, q_0), $(p_0 < \infty)$, and *strong* type (∞, q_1), then T is of joint weak type $(p_0, q_0; \infty, q_1)$. The relation (4.21) is replaced by the property that $\|f_1\|_\infty = f^*(t^m)$ and then the strong-type (∞, q_1) hypothesis is invoked in the form

$$T: L^\infty \to L^{q_1} \hookrightarrow L^{q_1,\infty},$$

so that the case $i = 1$ of (4.23) may be replaced by the estimate

$$(Tf_1)^*\left(\frac{t}{2}\right) \le \left(\frac{t}{2}\right)^{-1/q_1} M_1 \|f_1\|_\infty.$$

Now we can present the main result of this section, the Marcinkiewicz interpolation theorem. The following formulation in terms of Lorentz spaces is due to A. P. Calderón.

Theorem 4.13 (Marcinkiewicz interpolation theorem). *Suppose $1 \le p_0 < p_1 < \infty$ and $1 \le q_0, q_1 \le \infty$ with $q_0 \ne q_1$. Let $0 < \theta < 1$ and define p and q by*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{4.25}$$

Let T be a quasilinear operator defined on $(L^{p_0,1} + L^{p_1,1})(R,\mu)$ and taking values in $\mathcal{M}_0(S,\nu)$, where (R,μ) and (S,ν) are totally σ-finite measure spaces. Suppose T is of weak types (p_0, q_0) and (p_1, q_1), with respective weak-type norms M_0 and M_1. If $1 \le r \le \infty$, then

$$T: L^{p,r} \to L^{q,r}. \tag{4.26}$$

Precisely, there is a constant c, depending only on p_0, q_0, p_1, q_1, and r, such that, for all f in $L^{p,r}$,

$$\|Tf\|_{q,r} \le \frac{c}{\theta(1-\theta)} \max(M_0, M_1) \|f\|_{p,r}. \tag{4.27}$$

Proof. Note first that with m defined by (4.14) and p, q by (4.25), we have

$$\frac{1}{m}\left[\frac{1}{q} - \frac{1}{q_0}\right] = \frac{1}{p} - \frac{1}{p_0}, \qquad \frac{1}{m}\left[\frac{1}{q} - \frac{1}{q_1}\right] = \frac{1}{p} - \frac{1}{p_1}. \tag{4.28}$$

Because of Theorem 4.11, we know that T is of joint weak type $(p_0, q_0; p_1, q_1)$ with weak-type norm $M \le c \cdot \max(M_0, M_1)$, where c depends only on p_0, q_0, p_1, q_1. Hence, using (4.16) and (4.2), we have, in the case where $r < \infty$,

$$\|Tf\|_{q,r} \le M\left\{\int_0^\infty [t^{1/q} S_\sigma(f^*)(t)]^r \frac{dt}{t}\right\}^{1/r},$$

with S_σ defined as in (4.15). Applying Minkowski's inequality, we obtain

$$\|Tf\|_{q,r} \le M\left[\left\{\int_0^\infty \left[t^{1/q - 1/q_0} \int_0^{t^m} s^{1/p_0} f^*(s) \frac{ds}{s}\right]^r \frac{dt}{t}\right\}^{1/r} + \left\{\int_0^\infty \left[t^{1/q - 1/q_1} \int_{t^m}^\infty s^{1/p_1} f^*(s) \frac{ds}{s}\right]^r \frac{dt}{t}\right\}^{1/r}\right].$$

Making the change of variables $t^m = u$ in each of the integrals and using the relations (4.28), we therefore have

$$\|Tf\|_{q,r} \le M|m|^{-1/r}\left[\left\{\int_0^\infty\left[u^{1/p-1/p_0}\int_0^u s^{1/p_0}f^*(s)\frac{ds}{s}\right]^r\frac{du}{u}\right\}^{1/r}\right.$$
$$\left.+\left\{\int_0^\infty\left[u^{1/p-1/p_1}\int_u^\infty s^{1/p_1}f^*(s)\frac{ds}{s}\right]^r\frac{du}{u}\right\}^{1/r}\right].$$

When Hardy's inequalities III.(3.18) and III.(3.19) are applied to the first and second terms, respectively, the resulting estimate reduces to

$$\|Tf\|_{q,r} \le M|m|^{-1/r}\left[c_1\left\{\int_0^\infty[u^{1/p}f^*(u)]^r\frac{du}{u}\right\}^{1/r}\right.$$
$$\left.+c_2\left\{\int_0^\infty[u^{1/p}f^*(u)]^r\frac{du}{u}\right\}^{1/r}\right]$$
$$= M|m|^{-1/r}(c_1+c_2)\|f\|_{p,r},$$

with

$$\frac{1}{c_1} = \frac{1}{p_0} - \frac{1}{p} = \theta\left(\frac{1}{p_0} - \frac{1}{p_1}\right), \qquad \frac{1}{c_2} = \frac{1}{p} - \frac{1}{p_1} = (1-\theta)\left(\frac{1}{p_0} - \frac{1}{p_1}\right).$$

This establishes (4.27) for r finite. The proof for $r = \infty$ is similar. ■

The original theorem of Marcinkiewicz, formulated for Lebesgue spaces rather than the more general Lorentz spaces, is essentially as follows.

Corollary 4.14. *With parameters as above, suppose in addition that $p_i \le q_i$, $(i = 0, 1)$. If T is of weak types (p_0, q_0) and (p_1, q_1), with respective norms M_0 and M_1, then*

$$T: L^p \to L^q. \tag{4.29}$$

Precisely, there is a constant $c = c(p_0, q_0, p_1, q_1)$ such that

$$\|Tf\|_q \le \frac{c}{\theta(1-\theta)}\max(M_0, M_1)\|f\|_p \tag{4.30}$$

for all f in L^p.

Proof. Since $p_i \le q_i$, $(i = 0, 1)$, it follows from (4.25) that $p \le q$. Taking $r = p$ in (4.26), we conclude that

$$T: L^{p,p} = L^p \to L^{q,p}.$$

However, since $p \le q$, Proposition 4.2 shows that $L^{q,p}$ is continuously embedded in $L^{q,q} = L^q$, and from this the result follows. ■

Remarks 4.15 (a) Theorem 4.13 holds also in the case $p_1 = \infty$ provided the weak-type hypothesis "weak type (p_1, q_1)" is replaced by "strong type (p_1, q_1)". In that case (cf. Remark (4.12)(b)), the operator is of joint weak type $(p_0, q_0; \infty, q_1)$ and the proof of Theorem 4.13 carries over almost verbatim. As an example, consider the Hardy-Littlewood maximal operator M (Definition III.3.1), which is of weak type $(1, 1)$ (Theorem III.3.3) and of strong type (∞, ∞). Corollary 4.14 therefore establishes the L^p-boundedness $(1 < p < \infty)$ as asserted by the Hardy-Littlewood maximal theorem (Theorem III.3.10).
(b) More generally, Theorem 4.13 holds also in the case $p_1 = \infty$ when the separate weak-type hypotheses are replaced by the joint weak-type $(p_0, q_0; p_1, q_1)$ hypothesis. This is clear from an examination of the proof, which depends solely on the properties of the associated S_σ-operator. In this more general form, Theorem 4.13 may be regarded as a specialization to Lorentz $L^{p,q}$-spaces of the general Marcinkiewicz-type theorem for rearrangement-invariant spaces established in the previous chapter (Theorem III.5.7). The advantage of using the joint weak-type condition rather than the pair of individual weak-type conditions can be seen in connection with operators such as the Hilbert transform H (Definition III.4.1). This operator is of joint weak type $(1, 1; \infty, \infty)$ so, as we remarked above, the more general version of Corollary 4.14 for the joint weak-type operators can be applied to establish the L^p-boundedness of H for $1 < p < \infty$, as asserted by the M. Riesz theorem (Theorem III.4.9). The Hilbert transform is, of course, of weak type $(1, 1)$ (Theorem III.4.9) but not of strong type (∞, ∞), hence cannot be interpolated directly by the results above (one remedy is to use weak type $(1, 1)$ together with the strong-type $(2, 2)$ property of the Hilbert transform, so that Corollary 4.14 then yields the L^p-boundedness for $1 < p < 2$; a duality argument then produces the result for $2 < p < \infty$). It is clear that a meaningful notion of "weak type (∞, ∞)" would be desirable. We shall have more to say about this in Section V.7.
(c) The proof of the Marcinkiewicz interpolation theorem presented above for real spaces carries over in the obvious way for complex spaces, modulo an adjustment to the constant c in (4.27). The L^p-version (Corollary 4.14), producing a strong-type conclusion from weak-type hypotheses, is often more useful in the applications than the Riesz-Thorin theorem (Theorem 2.2). The additional restriction $p_i \le q_i$, $(i = 0, 1)$ in Corollary 4.14 rarely limits the applicability of the Marcinkiewicz theorem in practice.
(d) By a simple modification of the Calderón operator, the quantity