Analysis II
Special Functions

Defn The logarithm function is defined by

$$L(x) := \int_1^x 1/t\, dt, \quad x>0.$$

Properties

- $L(1) = 0$, $L'(x) = 1/x$, L strictly increasing and continuous.
- $L(x_1 x_2) = L(x_1) + L(x_2)$

  Pf: Use additivity of integration over subintervals, and change of variables.

  ### Corollaries

  - $L(x^n) = n L(x)$, $\lim_{x \to \infty} L(x) = \infty$

    Pf: Use induction with the previous result.
  - $L(1/x) = - L(x)$, $\lim_{x \to 0} L(x) = - \infty$

    Pf: Notice that $L(x\ 1/x) = 0$.
- $\mathrm{Dom}(L) = \mathbf{R}^+$, $\mathrm{Range}(L) = \mathbf{R}$.

  Pf: Since L is continuous, by the *intermediate value theorem* each real number must be in the range.

Defn The real number **e** is defined as the unique number such that $L(e) = 1$.

This number is unique since L is 1:1 (which follows since L is strictly increasing).

- **e >2**

  Pf: Use the fact that L strictly increases, $L(e) = 1 > L(2)$.
- $(L')'(x) = -1/x^2 < 0$; L' decreases and L is concave down.

Defn The logarithm function with base a>0 is defined by $\log_a(x) = L(x)/L(a)$.

This function satisfies all the properties of logarithm above, but with $\log_a(a) = 1$..

Defn The exponential function is defined as the inverse function of L(x), i.e. $E(x) = L^{-1}(x)$.

Properties

- $E(0) = 1$, $E(x)$ is a strictly increasing function with domain **R** and range $\mathbf{R}^+$.
- E is continuous and differentiable with $E'(x) = E(x)$.
  Pf: Use the formula to compute derivatives of inverse functions: $g'(x) = 1/(f'(g(x))$, if $g=f^{-1}$.
- $E(x_1 + x_2) = E(x_1)\, E(x_2)$, $E(mx) = E(x)^m$.
  Pf: Use the fact that $E(x) = y$ if and only if $L(y) = x$. Set $y_j = E(x_j)$, $j=1,2$.
- $E(1/n) = e^{(1/n)}$, $E(m/n) = e^{(m/n)}$ (for each rational *m/n*).
  Note: This allows us to extend e^x to all real numbers as $E(x)$ by using the fact that the rationals are dense and E is continuous.
  Pf: Let $y = E(1/n)$, then $y^n = .E(n\ 1/n) = e$. So y is the n-th root of e. The second statement follows as before by induction.
- $(E')'(x) = E(x)>0$, so E' strictly increases and E is concave up.

Defn The exponential function $a^x = E\big(x\, L(a)\big)$ has all the anticipated properties and is the inverse function of $\log_a(x)$.

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