ANALYSIS II
Introduction to Riemann-Stieltjes Integration

Defn. A collection of n+1 distinct points of the interval [a,b]

$$P := \{x_0 := a < x_1 < \ldots < x_{i-1} < x_i < \ldots < b =: x_n\}$$

is called a ***partition*** of the interval. In this case, we define the ***norm*** of the partition by

$$\|P\| := \max_{1 \le i \le n} \Delta x_i.$$

where $\Delta x_i := x_i - x_{i-1}$ is the ***length*** of the i-th subinterval $[x_{i-1}, x_i]$.
For a non-decreasing function α on [a,b], define

$$\Delta\alpha_i := \alpha(x_i) - \alpha(x_{i-1}).$$

Defn. Suppose that f is a bounded function on [a,b] and α is nondecreasing. For a given partition P, we define the ***Riemann-Stieltjes upper sum of a function f with respect to α*** by

$$U(P;f,\alpha) := \sum_{i=1}^{n} M_i \,\Delta\alpha_i$$

where M_i denotes the supremum of f over each of the subintervals $[x_{i-1}, x_i]$. Similarly, we define the *Riemann-Stieltjes lower sum* by

$$L(P;f,\alpha) := \sum_{i=1}^{n} m_i \,\Delta\alpha_i$$

where here m_i denotes the infimum of f over each of the subintervals $[x_{i-1}, x_i]$. Since $m_i \le M_i$ and $\Delta\alpha_i$ is nonnegative, we observe that

$$L(P;f,\alpha) \le U(P;f,\alpha).$$

for any partition P.

Defn. Suppose P_1, P_2 are both partitions of [a,b], then P_2 is called a ***refinement*** of P_1 (denoted by $P_1 \le P_2$) if as sets $P_1 \subseteq P_2$.

Note. If $P_1 \le P_2$, it follows that $\|P_2\| \le \| P_1\|$ since each of the subintervals formed by P_2 is contained in a subinterval which arises from P_1.

Lemma. If $P_1 \le P_2$, then

$$L(P_1;f,\alpha) \le L(P_2;f,\alpha).$$

and

$$U(P_2;f,\alpha) \le U(P_1;f,\alpha).$$

Pf. Suppose first that P_1 is a partition of [a,b] and that P_2 is the partition obtained from P_1 by adding an additional point z. The general case follows by induction, adding one point at a time. In particular, we let

$$P_1 := \{x_0 := a < x_1 < \ldots < x_{i-1} < x_i < \ldots < b =: x_n\}$$

and

$$P_2 := \{x_0 := a < x_1 < \ldots < x_{i-1} < z < x_i < \ldots < b =: x_n\}$$

for some fixed i. We focus on the upper sum for these two partitions, noting that the inequality for the lower sums follows similarly. Observe that

$$U(P_1;f,\alpha) := \sum_{j=1}^{n} M_j \, \Delta\alpha_j$$

and

$$U(P_2;f,\alpha) := \sum_{j=1}^{i-1} M_j \, \Delta\alpha_j + M\,(\alpha(z)-\alpha(x_{i-1})) + \tilde{M}\,(\alpha(x_i)-\alpha(z)) + \sum_{j=i+1}^{n} M_j \, \Delta\alpha_j$$

where $M := \sup \{ f(x) \mid x_{i-1} \le x \le z \}$ and $\tilde{M} := \sup \{ f(x) \mid z \le x \le x_i \}$. It then follows that $U(P_2;f,\alpha) \le U(P_1;f,\alpha)$ since

$$M, \tilde{M} \le M_i. \quad \Box$$

Defn. If P_1 and P_2 are arbitrary partitions of [a,b], then the ***common refinement*** of P_1 and P_2 is the formal union of the two.

Corollary. Suppose P_1 and P_2 are arbitrary partitions of [a,b], then

$$L(P_1;f,\alpha) \le U(P_2;f,\alpha).$$

Pf. Let P be the common refinement of P_1 and P_2, then

$$L(P_1;f,\alpha) \le L(P;f,\alpha) \le U(P;f,\alpha) \le U(P_2;f,\alpha). \quad \Box$$

Defn. The ***lower Riemann-Stieltjes integral of f with respect to α*** over [a,b] is defined to be

$$(L)\text{-}\int_a^b f(x)\, d\,\alpha := \sup_{\text{all partitions } P \text{ of } [a,b]} L(P;f,\alpha) .$$

Similarly, the ***upper Riemann-Stieltjes integral of f with respect to α*** over [a,b] is defined to be

$$(U)\text{-}\int_a^b f(x)\, d\,\alpha(x) := \inf_{\text{all partitions of } [a,b]} U(P;f,\alpha) .$$

By the definitions of least upper bound and greatest lower bound, it is evident that for any function f there holds

$$(L)\text{-}\int_a^b f(x)\, d\,\alpha(x) \;\leq\; (U)\text{-}\int_a^b f(x)\, d\,\alpha(x)\,.$$

Defn. A function f is ***Riemann-Stieltjes integrable*** over [a,b] if the upper and lower Riemann-Stieltjes integrals coincide. We denote this common value by $\int_a^b f(x)\, d\,\alpha(x)$.

Examples:

1. Obviously, if $\alpha(x) := x$, then the Riemann-Stieltjes integral reduces to the Riemann integral of f.
2. $\int_a^b f(x)\ d\,\alpha(x) = f(x_0)$, if f is continuous at x_0 and α is defined to be the step function which is one for x larger than x_0 and zero otherwise.

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