Analysis II

Riemann-Stieltjes Integration: Additional Results

Theorem. Suppose that f and α are both continuous and non-decreasing, then

$$\int_a^b f\, d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha\, df.$$

Proof. First we observe both integrals exist by our previous results. Let P be a partition

$$P: = \{x_0 : = a < x_1 < \dots < x_{i-1} < x_i < \dots < b = : x_n\}$$

and set $\xi_i = x_{i-1}$, $\xi'_i = x_i$, then

$$R(P;\xi,f,\alpha) = f(b)\alpha(b) - f(a)\alpha(a) - R(P;\xi',\alpha,f).$$

This follows from the ``summation by parts'' equation

$$\sum_{j=1}^{n} u_j (v_j - v_{j-1}) = u_n v_n - u_0 v_0 - \sum_{j=1}^{n} v_{j-1}(u_j - u_{j-1})$$

We set $u_j = f(x_j)$ and $v_j = \alpha(x_j)$ and chose a partition P so that both the left hand side is arbitrarily close to $\int_a^b f\, d\alpha$ and the Riemann-Stieltjes sum on the right hand side is arbitrarily close to $\int_a^b \alpha df$. $\square$

Theorem. If α is monotone increasing on [a,b] and f is bounded, then f is Riemann-Stieltjes integrable with respect to α on[a,b] if and only if $f\,\alpha'$ is Riemann integrable. In this case,

$$\int_a^b f\, d\alpha = \int_a^b f\alpha' dx$$

Pf. Suppose $\varepsilon > 0$ is arbitrary. Since α' is Riemann integrable, pick a partition P such that $U(P,\alpha')-L(P,\alpha') < \varepsilon$. For each subinterval $I_i: = [x_i,x_{i-1}]$, by the Mean Value Theorem there is a t_i in the interval so that $\Delta\alpha_i = \alpha'(t_i)\,\Delta x_i$. If s_i is arbitrary in I_i, then

$$\sum_{1}^{n} f(s_i)\, \Delta\alpha_i \quad = \quad \sum_{1}^{n} f(s_i)\, \alpha'(t_i)\, \Delta x_i$$

$$\leq \sum_{1}^{n} |f(s_i)|\, |\alpha'(t_i) - \alpha'(s_i)|\, \Delta x_i + U(P; f\,\alpha')$$

$$\leq \|f\|_\infty\, (U(P,\alpha') - L(P,\alpha')) + U(P; f\,\alpha')$$

$$\leq \|f\|_\infty\, \varepsilon + U(P; f\,\alpha').$$

By varying s_i, this implies

$$U(P;f,\alpha) - U(P;f\,\alpha') \leq \|f\|_\infty\, \varepsilon.$$

Using the same inequality, this estimate also holds for $U(P;f\,\alpha') - U(P;f,\alpha)$. A similar argument establishes the corresponding estimate for lower sums. □

Theorem. Suppose that f has range [m,M] and ϕ is continuous on [m,M]. If f is Riemann-Stieltjes integrable with respect to α, then $F = \phi \circ f$ is Riemann-Stiletjes integrable with respect to α.

Pf. Let $\varepsilon > 0$ be arbitary. Since ϕ is uniformly continuous, there is a $\delta > 0$ (which we may as well assume is smaller than ε) such that $|\phi(u)-\phi(v)|<\varepsilon$ if $|u-v| < \delta$. For this δ pick a partition P so that

$$U(P;f,\alpha) - L(P;f,\alpha) < \delta^2.$$

Let M_j, m_j denote sup and inf of f respectively over the subinterval $[x_{j-1},x_j]$. Similarly, define M_j^*, m_j^* as the sup and inf of F over same subinterval. Let G be defined as the index set of ``good'' intervals where $M_j - m_j < \delta$ and B as the remainder where $M_j - m_j \geq \delta$. For $j \in G$ we have $M_j^* - m_j^* < \varepsilon$, while for $j \in B$ there holds $\sum_{j \in B} \Delta\alpha_j < \delta$. The last estimate follows since $\delta \sum_{j \in B} \Delta\alpha_j \leq \sum_{j \in B} (M_i - m_i)\, \Delta\alpha_j \leq \delta^2$. Now we can estimate the difference between the upper and lower Riemann-Stieltjes sums of F:

$$U(P;F,\alpha) - L(P;F,\alpha) \leq \sum_{j \in G} (M_j^* - m_j^*)\, \Delta\alpha_j + \sum_{j \in B} (M_j^* - m_j^*)\, \Delta\alpha_j$$

$$\leq \varepsilon\, [\alpha(b)-\alpha(a)] + 2\, \|F\|_\infty \sum_{j \in B} \Delta\alpha_j$$

$$\leq \varepsilon\, [\alpha(b)-\alpha(a)] + 2\, \|F\|_\infty\, \delta \leq C\, \varepsilon. \quad \square$$

Corollary. If f is Riemann-Stieltjes integrable with respect to α, then so is f^2. Further, if g is also Riemann-Stieltjes integrable with respect to α, then the product f g is as well.

Pf. Apply the above theorem with $\phi(y) = y^2$ to establish the first statement. To establish the second, use the identity

$$f\, g = ((f+g)^2 - f^2 - g^2)/2$$

and note that each term of the sum on the right hand side is Riemann-Stieltjes integrable with respect to α. $\square$

Corollary. If f is Riemann-Stieltjes integrable with respect to α, then so is $|f|$.

Pf. Apply the above theorem with $\phi(y) = |y|$. $\square$

Defn. A function γ is said to be of ***bounded variation*** if for any partition P of the interval [a,b],

$$\mathrm{Var}_a^{\,b}(\gamma) := \sup_{\text{partitions } P} t(P,\gamma)$$

is finite where $t(P,\gamma) := \Sigma_{j=1}^{n} |\Delta\gamma_j|$.

Theorem. A function γ is of bounded variation if and only if it can be decomposed as a difference ($\gamma = \beta - \alpha$) of two monotone nondecreasing functions.

Pf. First define $u^+ = \max(u,0)$ and $u^- = \min(u,0)$, then $u = u^+ + u^-$ and $|u| = u^+ - u^-$. For a partition P of [a,x] define

$$p(P,x) = \Sigma_{j=1}^{n} |\Delta\gamma_j|^+$$

and

$$n(P,x) = \Sigma_{j=1}^{n} |\Delta\gamma_j|^{-}.$$

It is clear that $\beta(x) := \sup_{\text{partitions } P \text{ of } [a,x]} p(P,x)$ and $\alpha(x) = \sup_{\text{partitions } P \text{ of } [a,x]} -n(P,x)$ are nondecreasing functions. Finish by showing that $\gamma = \beta - \alpha$. □

Note. Using this decomposition, one may now extend both the definition of the Riemann-Stieltjes integral and its properties from γ monotone nondecreasing to γ being a function of bounded variation:

$$\int_a^b f\, d\gamma := \int_a^b f\, d\beta - \int_a^b f\, d\alpha.$$

All the properties given in the previous lectures have their obvious analogues. We may also easily extend to the case of complex and vector valued functions with corresponding results.

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