ANALYSIS II

Riemann-Stieltjes Integration: Conditions for Existence

In the previous section, we saw that it was possible for α to be discontinuous but for the Reiemann-Stieltjes integral of f to still exist. The following example shows that the integral may not exist however, if both f and α are discontinuous at a point.

Example. Let $f = \alpha$ where f(x) is one for nonnegative x and zero otherwise. In this case, if *P* is any partition, $U(P;f,\alpha) = 1$, while $L(P;f,\alpha) = 0$. This shows that the Riemann-Stieltjes integral for this pair does not exist.

Theorem. A necessary and sufficient condition for f to be Riemann-Stieltjes integrable with respect to α is for each given $\varepsilon > 0$, that one can obtain a partition P of [a,b] such that

(*)

$$U(P;f,\alpha) - L(P;f,\alpha) < \varepsilon.$$

Pf. First we show that (*) is a sufficient condition. This follows immediately, since for each $\varepsilon > 0$ that there is a partition P such that (*) holds,

$$(U) \int_a^b f(x)\, d\,\alpha(x) \;-\; (L) \int_a^b f(x)\, d\,\alpha(x) \;\le\; U(P;f,\alpha) - L(P;f,\alpha) \;<\; \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, then the upper and lower Riemann-Stieltjes integrals of f must coincide.

To prove that (*) is a necessary condition for f to be Riemann integrable, we let $\varepsilon > 0$. By the definition of the upper Riemann-Stieltjes integral as a infimum of upper sums, we can find a partition P_1 of [a,b] such that

$$\int_a^b f(x)\, d\alpha(x) \;\le\; U(P_1;f,\alpha) \;<\; \int_a^b f(x)\, d\alpha(x) \;+ \varepsilon/2$$

Similarly, we have

$$\int_a^b f(x)\, d\alpha(x) \;- \varepsilon/2 \;<\; L(P_2;f,\alpha) \;\le\; \int_a^b f(x)\, d\alpha(x)\,.$$

Let P be a common refinement of P_1 and P_2, then subtracting the two previous inequalities implies,

$$U(P;f,\alpha) - L(P;f,\alpha) \le U(P_1;f,\alpha) - L(P_2;f,\alpha) < \varepsilon. \quad \square$$

Theorem. If f is continuous on [a,b], then f is Riemann-Stieltjes integrable with respect to α on[a,b].

Pf. We use the condition (*) to establish the proof. If $\varepsilon > 0$, we set $\varepsilon_0 := \varepsilon/(1+\alpha(b)-\alpha(a))$. Since f is continuous on [a,b], f is uniformly continuous. Hence there is a $\delta > 0$ such that $|f(y)-f(x)| < \varepsilon_0$ if $|y-x| < \delta$. Suppose that $\|P\| < \delta$, then it follows that $|M_i - m_i| < \varepsilon_0 \quad (1 \le i \le n)$. Hence

$$U(P;f,\alpha) - L(P;f,\alpha) = \sum_{i=1}^{n} (M_i - m_i)\,\Delta\alpha_i < \varepsilon_0\,(\alpha(b)-\alpha(a)) < \varepsilon. \quad \square$$

Theorem. If f is monotone and α is continuous on [a,b], then f is Riemann-Stieltjes integrable with respect to α on [a,b].

Pf. We prove the case for f monotone increasing and note that the case for monotone decreasing is similiar. We again use the condition (*) to prove the theorem. If $\varepsilon > 0$, we set $\varepsilon_0 := \varepsilon/(1+f(b)-f(a))$, Since α is continuous and [a,b] is compact, α is uniformly continuous. So for ε_0 we can determine a $\delta > 0$, so that if P is a partition with $\|P\| < \delta$, then $\Delta\alpha_i < \varepsilon_0$ (all i). The function f is monotone increasing on [a,b], so $M_i = f(x_i)$ and $m_i = f(x_{i-1})$. Hence

$$\begin{aligned} U(P;f,\alpha) - L(P;f,\alpha) &= \sum_{i=1}^{n} (M_i - m_i)\,\Delta\alpha_i \\ &= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\,\Delta\alpha_i \\ &< \varepsilon_0 \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \\ &\le \varepsilon_0\,(f(b)-f(a)) < \varepsilon. \quad \square \end{aligned}$$

Defn. A ***Riemann-Stieltjes sum*** for f with respect to α for a partition P of an interval [a,b] is defined by

$$R(P;\xi) := \sum_{j=1}^{n} f(\xi_j)\,\Delta\alpha_j$$

where the ξ_j, satisfying $x_{j-1} \le \xi_j \le x_j$ $(1 \le j \le n)$, are arbitrary.

Corollary. Suppose that f is Riemann-Stieltjes integrable on [a,b], then there is a unique number γ ($= \int_a^b f\, d\alpha$) such that for every $\varepsilon > 0$ there exists a partition P of [a,b] such that if $P \le P_1, P_2$, then

i.) $0 \le U(P_1;f,\alpha) - \gamma < \varepsilon$

ii.) $0 \le \gamma - L(P_2;f,\alpha) < \varepsilon$

iii.) $|\gamma - R(P_1,\xi)| < \varepsilon$

where $R(P_1,\xi)$ is any Riemann-Stieltjes sum of f with respect to α for the partition P_1. In this case, we can interpret the integral as

$$\int_a^b f\, d\alpha = \lim_{\|P\| \to 0} R(P,\xi),$$

although a careful proof is somewhat involved.

Pf. Since $L(P_2;f,\alpha) \le \gamma \le U(P_1;f,\alpha)$ for all partitions, we see that parts i.) and ii.) follow from the definition of the integral. To see part iii.), we observe that $m_j \le f(\xi_j) \le M_j$ and hence that

$$L(P_1;f,\alpha) \le R(P_1,\xi) \le U(P_1;f,\alpha).$$

But we also know that both

$$L(P_1;f,\alpha) \le \gamma \le U(P_1;f,\alpha)$$

and condition (*) hold, from which part iii.) follows. $\square$

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