ANALYSIS II
Metric Spaces: Limits and Continuity

Defn Suppose (X,d) is a metric space and A is a subset of X.

1. A point x is called an ***interior point*** of A if there is a neighborhood of x contained in A.
2. A set N is called a ***neighborhood (nbhd) of x*** if x is an interior point of N.
3. A point x is called a ***boundary point*** of A if it is a limit point of both A and its complement.
4. A point x is called a ***limit point*** of the set A if each neighborhood of x contains points of A distinct from x.
   (This is equivalent to saying that each neighborhood of x has an infinite number of members of A. Recall that a neighborhood for a point x, is a set containing an open ε-nbhd of x.)
5. A point x is called an ***isolated point*** of A if x belongs to A but is not a limit point of A.

Proposition A set O in a metric space is open if and only if each of its points are interior points.

Proposition A set C in a metric space is closed if and only if it contains all its limit points.

Defn Suppose (X,d) is a metric space and A is a subset of X. The ***closure of A*** is the smallest closed subset of X which contains A. The ***derived set A′*** of A is the set of all limit points of A.

Proposition The closure of A may be determined by either

- the intersection of all closed sets which contain A,

or

- the union of A with its derived set.

Sequential Convergence

Defn A sequence $\{x_n\}$ in a metric space (X,d) is said to ***converge***, to a point x_0 say, if for each neighborhood of x_0 there exists a natural number N so that x_n belongs to the neighborhood if n is greater or equal to N; that is, eventually the sequence is contained in the neighborhood. In this case, we say that x_0 is the ***limit of the sequence*** and write

$$\lim_{n \to \infty} x_n := x_0 .$$

Proposition In a metric space, sequential limits are unique.

Proposition That a sequence $\{x_n\}$ converges in a metric space (X,d) to a point x_0 is equivalent to the condition that for each $\epsilon > 0$ there is a natural number N such that $N \le n$ implies $d(x_n , x_0) < \epsilon$.

Examples

1. In either the reals or complexes if $|r| < 1$, then $r^n \to 0$.
2. Consider the space of continuous functions on [0,1/2], C[0,1/2]. Let $f_n(x) = x^n$, then $f_n \to 0$.
3. The sequence $f_n(x) = x^n$ belongs to C[0,1] but does not converge.

Defn A ***function f*** defined on $X \backslash \{x_0\}$, with values in a metric space $\{Y,d_2\}$ ***is said to have a limit L at x_0*** if x_0 is a limit point of X and for each neighborhood O_2 of L, there is a neighborhood O_1 of x_0 such that f maps each element of the deleted neighborhood $O_1 \backslash \{x_0\}$ into O_2 . This is denoted

$$\lim_{x \to x_0} f(x) := L.$$

Homework This is equivalent to the condition: for each $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < d_1(x,x_0) < \delta$, then $d_2(f(x),L) < \epsilon$.

Proposition A necessary and sufficient condition for a function f to have a limit L at x_0 is that for each sequence $\{x_n\}$ which converges to x_0 (no point of which is equal to x_0), then $\{f(x_n)\}$ converges to L. Consequently, if a function has a limit at a point x_0, then it is unique.

Defn A ***function f*** is called ***continuous at a point x_0*** if either

1. x_0 is an isolated point of X or
2. x_0 is a limit point of X and the limit of f as x approaches x_0 is $f(x_0)$.

Homework A necessary and sufficient condition for a function f to be continuous at x_0 is that for each $\epsilon > 0$ there is a $\delta > 0$ such that if $d_1(x,x_0) < \delta$, then $d_2(f(x),f(x_0)) < \epsilon$.

Continuity

Defn Suppose $f : X \to Y$ where (X,d_1) and (Y,d_2) are metric spaces. f is called ***continuous*** if the inverse image of each open set in Y is open in X.

Proposition A function $f : X \to Y$ is continuous if and only if the inverse image of each closed set in Y is closed in X.

Theorem A function $f : X \to Y$ is continuous if and only if f is continuous at each point of X.

Theorem Suppose that $f: X \to Y$ and $g: Y \to Z$ are continuous functions, then $g \circ f$ is a continuous function from X to Z.

Theorem Suppose that (X,d_X) and (Y,d_Y) are both metric spaces, then X x Y is a metric space if the metric d is defined for $z_i = (x_i,y_i)$, i=1,2, by

$$d(z_1,z_2) := d_X(x_1,x_2) + d_Y(y_1,y_2).$$

Examples:

1. For a metric space (X,d), the metric d is a continuous function from X^2 to R.
2. Suppose that $(X,\|.\|)$ is a normed linear space, then both the vector space operations are jointly continuous:
    1. if $a_n \to a$ in R and $x_n \to x$ in X, then $\|a_n x_n\|_X \to \|a x\|_X$ in R.
    2. if $x_n \to x$ and $y_n \to y$ in X, then $x_n + y_n \to x + y$ in X.

Robert Sharpley Jan 25 1998