ANALYSIS II
Metric Spaces: Connectedness

Defn. A ***disconnection*** of a set A in a metric space (X,d) consists of two nonempty sets A_1, A_2 whose disjoint union is A and each is open relative to A. A set is said to be ***connected*** if it does not have any disconnections.

Example. The set $(0,1/2) \cup (1/2,1)$ is disconnected in the real number system.

Theorem. Each interval (open, closed, half-open) I in the real number system is a connected set.

> Pf. Let A_1,A_2 be a disconnection for I. Let $a_j \in A_j$, $j = 1,2$. We may assume WLOG that $a_1 < a_2$, otherwise relabel A_1 and A_2. Consider $E_1 := \{x \in A_1 \mid x \le a_2\}$, then E_1 is nonempty and bounded from above. Let $a := \sup E_1$. But $a_1 \le a \le a_2$ implies $a \in I$ since I is an interval. First note that by the lemma to the least upper bound property either $a \in A_1$ or a is a limit point of A_1. In either case, $a \in A_1$ since A_1 is closed relative to I. Since A_1 is also open relative to the interval I, then there is an $\varepsilon > 0$ so that $N_\varepsilon(a) \in A_1$. But then $a+\varepsilon/2 \in A_1$ and is less than a_2, which contradicts that a is the sup of E_1. □

Theorem. If A is a connected subset of real numbers (with the standard metric), then A is an interval.

> Pf. Otherwise, there would be $a_1 < a < a_2$, with $a_j \in A$ and a does not belong to A. But then $O_1 := (-\infty,a) \cap A$ and $O_2 := (a,\infty) \cap A$ form a disconnection of A. □

Note. Each open subset of IR is the countable disjoint union of open intervals. This is seen by looking at open ***components*** (maximal connected sets) and recalling that each open interval contains a rational. Relatively (with respect to $A \subseteq$ IR) open sets are just restrictions of these.

Theorem. The continuous image of a connected set is connected.

> Pf. If C is a connected set in a metric space X and there is a disconnection of the image f(C), then it can be `drawn back' to form a disconnection of C : if $\{ O_1, O_2 \}$ forms a disconnection for f(C), then $\{ f^{-1}(O_1), f^{-1}(O_2) \}$ forms a disconnection for C. □

Corollary. (Intermediate Value Theorem) Suppose f is a real-valued function which is continuous on an interval I. If $a_1, a_2 \in I$ and y is a number between $f(a_1)$ and $f(a_2)$, then there exists a between a_1 and a_2 such that $f(a) = y$.

Pf. We may assume WLOG that $I = [a_1, a_2]$. We know that f(I) is a closed interval, say I_1. Any number y between $f(a_1)$ and $f(a_2)$, belongs to I_1 and so there is an $a \in [a_1,a_2]$ such that $f(a) = y$. □

Corollary. The continuous image of a closed and bounded interval [a,b] is an interval [c,d] where

$$c = \min_{a \leq x \leq b} f(x)$$

$$d = \max_{a \leq x \leq b} f(x).$$

Corollary. (Fixed Point Theorem) Suppose that $f:[a,b] \to [a,b]$ is continuous, then f has a fixed point, i.e. there is an $\alpha \in [a,b]$ such that $f(\alpha) = \alpha$.

Pf. Consider the function $g(x) := x - f(x)$, then $g(a) \leq 0 \leq g(b)$. g is continuous on [a,b], so by the Intermediate Value Theorem, there is an $\alpha \in [a,b]$ such that $g(\alpha) = 0$. This implies that $f(\alpha) = \alpha$. □

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