ANALYSIS II

Metric Spaces: Compactness

Defn A collection of open sets is said to be an ***open cover*** for a set A if the union of the collection contains A. A subset of an open cover whose union also contains the set A is called a ***subcover*** of the original cover. A cover is called ***finite*** if it has finitely many members.

Defn A set K in a metric space (X,d) is said to be ***compact*** if each open cover of K has a finite subcover.

Theorem Each compact set K in a metric space is closed and bounded.

Proposition Each closed subset of a compact set is also compact.

Theorem (Heine-Borel Theorem from last term) Each closed and bounded interval [a,b] is a compact subset of the real numbers.

Pf: Let $\mathcal{C}$ be an open cover for [a,b] and consider the set A := $\{x \mid [a,x]$ has an open cover from $\mathcal{C}\}$. Note that A is not empty since a belongs to A. Let c:= lub (A). It is enough to show that $c > b$, since if x_1 belongs to A and $a \le x \le x_1$, then x belongs to A. Suppose instead that $c \le b$, then there must be some O_0 in C such that c belongs to O_0. But O_0 is open, so there exists $\delta > 0$ so that $N_\delta(c)$ is contained in O_0. Since c is the least upper bound for A, then there is an x in A such that $c-\delta < x \le c$. But x belongs to A so there are members $O_1, \ldots, O_n$ of $\mathcal{C}$ whose union covers [a,x]. The collection $O_1, \ldots, O_n$ covers $[a, c+\delta/2]$. **Contradiction**, since c is the least upper bound for the set A.

Corollary Each closed and bounded set of real numbers is compact.

Theorem If a set A is compact in a metric space X and f: $X \to Y$ is continuous, then f[A] is compact in Y.

Corollary If $f : X \to Y$ is continuous and X is compact, then f is a bounded function.

Corollary If $f : X \to \mathbf{R}$ is continuous and X is compact, then f attains its extremal values.

Theorem Suppose that f: $[a,b] \to K$ is one-to-one, onto and continuous, then f^{-1} is continuous.

Pf. Let $O \subseteq [a,b]$ be relatively open, then $(f^{-1})^{-1}(O) = f(O)$. Let C be the complement in [a,b] of O, then C is closed and hence compact. Therefore f(C) is compact in K and conseqently it is closed. Its complement in K must then be relatively open. That complement however is f(O). □

Compactness Characterization Theorem Suppose that K is a subset of a metric space X, then the following are equivalent:

1. K is compact.
2. each infinite subset of K has a limit point in K.
3. each sequence from K has a subsequence that converges in K.

(Click here for the details of the proofs.)

Corollary Each closed and bounded set K in R^k (or C^k) is compact.

Pf: Use the sequential convergence criterium and consider projections into each coordinate. Recall that convergence in R^k is equivalent to convergence in each coordinate.

Defn A set K in a metric space X is said to be ***totally bounded***, if for each $\epsilon > 0$ there are a finite number of open balls with radius ϵ which cover K. Here the centers of the balls and the total number will depend in general on ϵ.

Theorem A set K in a metric space is compact if and only if it is complete and totally bounded. [**Homework**.]

Robert Sharpley Feb 14 1998