ANALYSIS II
Proof of Compactness Characterizatons

Compactness Characterization Theorem
Suppose that K is a subset of a metric space X, then the following are equivalent:

1. K is compact,
2. K satisfies the Bolzanno-Weierstrass property (i.e., each infinite subset of K has a limit point in K),
3. K is sequentially compact (i.e., each sequence from K has a subsequence that converges in K).

Defn A set K in a metric space X is said to be ***totally bounded***, if for each $\epsilon > 0$ there are a finite number of open balls with radius ϵ which cover K. Here the centers of the balls and the total number will depend in general on ϵ.

Defn A set D is said to be ***dense*** in a set A if each neighborhood of each point x of A contains a member from D. A set A in a metric space is called ***separable*** if it has a countable dense subset.

(Compactness → the Bolzanno-Weierstrass property)
Suppose K is compact, but that A is an infinite subset of K with no limit point in K. But K is closed since it is compact, so the derived set of A is empty and A is therefore closed. In particular, each point of A must be isolated. Hence for each x in A there is a ball B_x which contains only a single member from A (namely x). The collection of these balls (together with the complement of A),

$$\{B_x \mid x \text{ belongs to } A\} \cup \{A^c\}$$

is an open cover for K. In any finite subcover (which must exist by the compactness of K), one of the balls must contain an infinite number of elements of A since A is infinite. **Contradiction**, since each one of these balls has exactly one member from A.

(Bolzanno-Weierstrass property → Sequential Compactness)
Suppose $\{x_n\}_n$ is a sequence in K and K has the Bolzanno-Weierstrass property. If the range of the sequence is finite, we are done since one of the values must be repeated infinitely often and then that subsequence converges since it is a constant sequence. In the case the range of the sequence is infinite, we apply the B-W property to obtain a limit point x_0 in K. A subsequence is

then easily constructed which converges to x_0 .

(Sequential Compactness → Completeness) [**Homework**]

(Sequential Compactness → Totally Bounded) [**Homework**]
(Hint: Suppose not, then by induction there exists $\delta > 0$ and a sequence of points $x_1, \ldots, x_n, \ldots$ such that x_{n+1} does not belong to the $B_\delta(x_0) \cup \ldots \cup B_\delta(x_n)$. This will give rise to a contradiction since all members of the sequence are at least δ units apart and there can be no convergent subsequence.)

(Sequential Compactness → separable and that the topology has a countable base)
K is totally bounded so there are a finite number of balls $B_1(x_{1,1}), \ldots, B_1(x_{1,n(1)})$, of radius 1 which cover K. Continuing with each natural number k, we find a finite number of balls $B_{1/k}(x_{k,1}), \ldots, B_{1/k}(x_{k,n(k)})$ which cover K. The collection of all centers then forms a countable dense subset of K. [**Homework: Prove this**] Moreover, for each point x_0 in K and each neighborhood $\mathcal{O}$ of x_0 , there is a ball from this countable list containing x_0 which is contained in $\mathcal{O}$, i.e. the topology of K has a \`***countable base***'. Notice, in fact, that each open set can be written as a union of this countable collection of open balls.

(Sequential Compactness → each open cover of K has a countable subcover)
Let $\mathcal{C}$ be an open cover of K. Each point x in K belongs to some $\mathcal{O}_x$ in $\mathcal{C}$. There is a countable base (of balls) $\{B_n\}_n$ for the topology so there exists a natural number n(x) so that x belongs to $B_{n(x)}$ which is contained in $\mathcal{O}_x$. For each of these balls B_n which arise in this way, select one such $\mathcal{O}_x$ and (abusing our notation and thereby our sensibilities) label it $\mathcal{O}_n$. This constructs the desired countable subcover of $\mathcal{C}$.

(Sequential Compactness → Compactness)
Let $\mathcal{C}$ be an open cover for a sequentially compact set K. Let $\{\mathcal{O}_1 , \ldots, \mathcal{O}_n , \ldots\}$ be a countable subcover of $\mathcal{C}$ which covers K. If $\{\mathcal{O}_1 , \ldots, \mathcal{O}_k \}$ is not a finite subcover of K, then there is a member of K, x_{k+1} say, which lies outside the union of this finite collection. By sequentially compactness, there is a subsequence $\{y_j\}$ of $\{x_k\}$ which converges to some y_0 in K. **Contradiction**, since y_0 will be in one of these open sets and by the definition of convergence, all but a finite number of the y_j's will also belong to this same set. But this is impossible by our construction.

Note If one looks closely, we have in fact shown another important characterization of compactness in metric spaces:

Theorem   A set K in a metric space is compact if and only if it is complete and totally bounded.

[**Extra Credit Homework**]

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