Math 554 – Integration
Handout #9 – 4/12/96

Defn. A collection of $n+1$ distinct points of the interval $[a,b]$

$$P := \{x_0 = a < x_1 < \cdots < x_{i-1} < x_i < \cdots < b =: x_n\}$$

is called a *partition* of the interval. In this case, we define the *norm* of the partition by

$$\|P\| := \max_{1\le i\le n} \Delta x_i.$$

where $\Delta x_i := x_i - x_{i-1}$ is the *length* of the i-th subinterval $[x_{i-1}, x_i]$.

Defn. For a given partition P, we define the *Riemann upper sum* of a function f by

$$U(P,f) := \sum_{i=1}^{n} M_i \, \Delta x_i$$

where M_i denotes the supremum of f over each of the subintervals $[x_{i-1}, x_i]$. Similarly, we define the *Riemann lower sum* of a function f by

$$L(P,f) := \sum_{i=1}^{n} m_i \, \Delta x_i$$

where m_i denotes the infimum of f over each of the subintervals $[x_{i-1}, x_i]$. Since $m_i \le M_i$, we note that

$$L(P,f) \le U(P,f).$$

for any partition P.

Defn. Suppose P_1, P_2 are both partitions of $[a,b]$, then P_2 is called a *refinement of* P_1, denoted by

$$P_1 \prec P_2,$$

if as sets $P_1 \subseteq P_2$.

Note. If $P_1 \prec P_2$, it follows that $\|P_2\| \le \|P_1\|$ since each of the subintervals formed by P_2 is contained in a subinterval which arises from P_1.

Lemma. If $P_1 \prec P_2$, then

$$L(P_1,f) \le L(P_2,f).$$

and

$$U(P_2,f) \le U(P_1,f).$$

Proof. Suppose first that P_1 is a partition of $[a, b]$ and that P_2 is the partition obtained from P_1 by adding an additional point z. The general case follows by induction, adding one point at at time. In particular, we let

$$P_1 := \{x_0 = a < x_1 < \cdots < x_{i-1} < x_i < \cdots < b =: x_n\}$$

and

$$P_2 := \{x_0 = a < x_1 < \cdots < x_{i-1} < z < x_i < \cdots < b =: x_n\}$$

for some fixed i. We focus on the upper Riemann sum for these two partitions, noting that the inequality for the lower sums follows similarly. Observe that

$$U(P_1, f) := \sum_{j=1}^{n} M_j \, \Delta x_j$$

and

$$U(P_2, f) := \sum_{j=1}^{i-1} M_j \, \Delta x_j + M(z - x_{i-1}) + \tilde{M}(x_i - z) + \sum_{j=i+1}^{n} M_j \, \Delta x_j$$

where $M := \sup_{[x_{i-1}, z]} f(x)$ and $\tilde{M} := \sup_{[z, x_i]} f(x)$. It then follows that $U(P_2, f) \leq U(P_1, f)$ since

$$M, \tilde{M} \leq M_i. \quad \square$$

Defn. If P_1 and P_2 are arbitrary partitions of $[a, b]$, then the *common refinement* of P_1 and P_2 is the formal union of the two.

Corollary. Suppose P_1 and P_2 are arbitrary partitions of $[a, b]$, then

$$L(P_1, f) \leq U(P_2, f).$$

Proof. Let P be the common refinement of P_1 and P_2, then

$$L(P_1, f) \leq L(P, f) \leq U(P, f) \leq U(P_2, f). \quad \square$$

Defn. The *lower Riemann integral* of f over $[a, b]$ is defined to be

$$\underline{\int_a^b} f(x)dx := \sup_{\substack{\text{all partitions} \\ \text{P of [a,b]}}} L(P, f).$$

Similarly, the *upper Riemann integral* of f over $[a, b]$ is defined to be

$$\overline{\int_a^b} f(x)dx := \inf_{\substack{\text{all partitions} \\ \text{P of [a,b]}}} U(P, f).$$

By the definitions of least upper bound and greatest lower bound, it is evident that for any function f there holds

$$\underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx.$$

Defn. A function f is *Riemann integrable over* $[a, b]$ if the upper and lower Riemann integrals coincide. We denote this common value by $\int_a^b f(x)\,dx$.

Examples: 1. $\int_a^b k\,dx = k(b-a)$.

2. $\int_a^b x\,dx = \frac{1}{2}(b^2 - a^2)$.
[Hint: Use $\sum_{i=1}^n i = n(n+1)/2$.]

Theorem. A necessary and sufficient condition for f to be Riemann integrable is given $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

$$(*) \qquad U(P, f) - L(P, f) < \epsilon.$$

Proof. First we show that (*) is a sufficient condition. This follows immediately, since for each $\epsilon > 0$ that there is a partition P such that (*) holds,

$$\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \le U(P, f) - L(P, f) < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, then the upper and lower Riemann integrals of f must coincide.

To prove that (*) is a necessary condition for f to be Riemann integrable, we let $\epsilon > 0$. By the definition of the upper Riemann integral as a infimum of upper sums, we can find a partition P_1 of $[a, b]$ such that

$$\int_a^b f(x)dx \le U(P_1, f) < \int_a^b f(x)dx + \epsilon/2$$

Similarly, we have

$$\int_a^b f(x)dx - \epsilon/2 < L(P_2, f) \le \int_a^b f(x)dx.$$

Let P be a common refinement of P_1 and P_2, then subtracting the two previous inequalities implies,

$$U(P, f) - L(P, f) \le U(P_1, f) - L(P_2, f) < \epsilon. \qquad \square$$

Defn. A *Riemann sum* for f for a partition P of an interval $[a, b]$ is defined by

$$R(P, f, \boldsymbol{\xi}) := \sum_{j=1}^n f(\xi_j)\Delta x_j$$

where the ξ_j, satisfying $x_{j-1} \leq \xi_j \leq x_j$ $(1 \leq j \leq n)$, are arbitrary.

Corollary. Suppose that f is Riemann integrable on $[a, b]$, then there is a unique number γ ($= \int_a^b f(x)dx$) such that for every $\epsilon > 0$ there exists a partition P of $[a, b]$ such that if $P \prec P_1, P_2$, then

i.) $\quad 0 \leq U(P_1, f) - \gamma < \epsilon$

ii.) $\quad 0 \leq \gamma - L(P_2, f) < \epsilon$

iii.) $\quad |\gamma - R(P_1, f, \boldsymbol{\xi})| < \epsilon$

where $R(P_1, f, \boldsymbol{\xi})$ is any Riemann sum of f for the partition P_1. In this sense, we can interpret

$$\int_a^b f(x)dx = \lim_{\|P\| \to 0} R(P, f, \boldsymbol{\xi}).$$

although we would actually need to show a little more to be entirely correct.

Proof. Since $L(P_2, f) \leq \gamma \leq U(P_1, f)$ for all partitions, we see that parts i.) and ii.) follow from the definition of the Riemann integral. To see part iii.), we observe that $m_j \leq f(\xi_j) \leq M_j$ and hence that

$$L(P_1, f) \leq R(P_1, f, \boldsymbol{\xi}) \leq U(P_1, f).$$

But we also know that both

$$L(P_1, f) \leq \gamma \leq U(P_1, f)$$

and condition (*) hold, from which part iii.) follows. $\square$

Theorem. If f is continuous on $[a, b]$, then f is Riemann-integrable on $[a, b]$.
Proof. We use the condition (*) to prove that f is Riemann-integrable. If $\epsilon > 0$, we set $\epsilon_0 := \epsilon/(b-a)$. Since f is continuous on $[a, b]$, f is uniformly continuous. Hence there is a $\delta > 0$ such that $|f(y) - f(x)| < \epsilon_0$ if $|y - x| < \delta$. Suppose that $\|P\| < \delta$, then it follows that $|M_i - m_i| \leq \epsilon_0 \quad (1 \leq i \leq n)$. Hence

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \leq \epsilon_0(b-a) = \epsilon. \quad \square$$

Theorem. If f is monotone on $[a, b]$, then f is Riemann-integrable on $[a, b]$.
Proof. If f is constant, then we are done. We prove the case for f monotone increasing. The case for monotone decreasing is similiar. We again use the condition (*) to prove that f is Riemann-integrable. If $\epsilon > 0$, we set $\delta := \epsilon/(f(b) - f(a))$ and consider any partition P with $\|P\| < \delta$. Since f is monotone increasing on $[a, b]$, then $M_i = f(x_i)$ and $m_i = f(x_{i-1})$. Hence

$$\begin{aligned} U(P, f) - L(P, f) &= \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \\ &= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\Delta x_i \\ &\leq \|P\| \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \\ &< \delta\,(f(b) - f(a)) = \epsilon. \qquad \square \end{aligned}$$

Theorem. (Properties of the Riemann Integral) Suppose that f and g are Riemann integrable and k is a real number, then

i.) $\int_a^b k\, f(x)\, dx = k\, \int_a^b f(x)\, dx$

ii.) $\int_a^b f + g\, dx = \int_a^b f\, dx + \int_a^b g\, dx$

iii.) $g \leq f$ implies $\int_a^b g\, dx \leq \int_a^b f\, dx$.

iv.) $|\int_a^b f\, dx| \leq \int_a^b |f|\, dx$

Proof. To prove part i.), we observe that in case $k \geq 0$, then $sup_{[x_{i-1}, x_i]} kf(x) = kM_i$ and $inf_{[x_{i-1}, x_i]} kf(x) = km_i$. Hence $U(P, kf) = kU(P, f)$ and $L(P, kf) = kL(P, f)$. In the case that $k < 0$, then $sup_{[x_{i-1}, x_i]}\, kf(x) = km_i$ and $inf_{[x_{i-1}, x_i]}\, kf(x) = kM_i$. It follows in this case that $U(P, kf) = kL(P, f)$ and $L(P, kf) = kU(P, f)$ and so

$$\overline{\int_a^b} k\, f(x)dx = k \underline{\int_a^b} f(x)dx$$

$$\underline{\int_a^b} k\, f(x)dx = k \overline{\int_a^b} f(x)dx.$$

To prove property ii.) we notice that $\sup_I(f+g) \leq \sup_I f + \sup_I g$ and $\inf_I f + \inf_I g \leq \inf_I(f+g)$ for any interval I (for example, $I = [x_{i-1}, x_i]$). Hence,

$$(1) \qquad L(P,f) + L(P,g) \leq L(P, f+g) \leq U(P, f+g) \leq U(P,f) + U(P,g).$$

Let $\epsilon > 0$, then since f and g are Riemann integrable, there exist partitions P_1, P_2 such that

$$(2) \qquad U(P_1,f) - L(P_1,f) < \epsilon/2, \quad U(P_2,g) - L(P_2,g) < \epsilon/2.$$

If we let P be a common refinement of P_1 and P_2, then by combining inequalities (1) and (2), we see that see that

$$\begin{aligned} U(P, f+g) - L(P, f+g) &\leq U(P,f) - L(P,f) \ + \ U(P,g) - L(P,g) \\ &\leq U(P_1,f) - L(P_1,f) \ + \ U(P_2,g) - L(P_2,g) \\ &\leq \epsilon/2 \ + \ \epsilon/2 = \epsilon. \end{aligned}$$

Property iii.) follows directly from the definition of the upper and lower integrals using, for example, the inequality $sup_I\, g(x) \leq sup_I\, f(x)$.

Property iv.) is proved by applying property iii.) to the inequality

$$-|f| \leq f \leq |f|,$$

from which it follows that $-\int_a^b |f|\, dx \leq \int_a^b f\, dx \leq \int_a^b |f|\, dx$. But this inequality implies property iv.). □

Defn. We extend the definition of the integral to include general limits of integration. These are consistent with our earlier definition.

1. $\int_a^a f(x)\, dx = 0$.
2. $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$.

Theorem. If f is Riemann integrable on $[a,b]$, then it is Riemann integrable on each subinterval $[c,d] \subseteq [a,b]$. Moreover, if $c \in [a,b]$, then

$$(3) \qquad \int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

Proof. We show first that condition (*) holds for the interval $[c,d]$. Suppose $\epsilon > 0$, then by (*) applied to f over the interval $[a,b]$, we have that there exists a partition P of $[a,b]$ such that condition (*) holds. Let $\tilde{P}$ be the refinement obtained from P which contains the points c and d. Let P^* be the partition obtained by restricting the partition $\tilde{P}$ to the interval $[c,d]$, then

$$U(P^*,f) - L(P^*,f) \le U(\tilde{P},f) - L(\tilde{P},f) \le U(P,f) - L(P,f) < \epsilon$$

and so f is Riemann integrable over $[c,d]$.

To prove the identity (3), we use the fact that condition (*) holds when f is Riemann integrable. Let $\epsilon > 0$, then for $\epsilon/3 > 0$, we may apply (*) to each of the intervals $I = [a,b], [a,c]$ and $[c,b]$, respectively, to obtain partitions P_I which satisfy

$$0 \le U_I(P_I,f) - \int_I f\,dx \le U_I(P_I,f) - L_I(P_I,f) < \epsilon/3. \tag{4}$$

We let P be the partition of $[a,b]$ formed by the union of the two partitions $P_{[a,c]}, P_{[c,b]}$, and $\tilde{P}$ be the common refinement of P and $P_{[a,b]}$. Observing that

$$U_{[a,b]}(\tilde{P},f) = U_{[a,c]}(\tilde{P}_1,f) + U_{[c,b]}(\tilde{P}_2,f), \tag{5}$$

we can combine with inequality (4) to obtain

$$\begin{aligned} \left|\textstyle\int_a^c f\,dx + \int_c^b f\,dx - \int_a^b f\,dx\right| \ &\le \ \left|U_{[a,c]}(\tilde{P},f) - \textstyle\int_a^c f\,dx\right| + \left|U_{[c,b]}(\tilde{P},f) - \textstyle\int_c^b f\,dx\right| \\ &\qquad + \left|U_{[a,b]}(\tilde{P},f) - \textstyle\int_a^b f\,dx\right| \\ &< \ 3\epsilon_0 = \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, then equality (3) must hold. $\square$

Theorem. (Intermediate Value Theorem for Integrals) If f is continuous on $[a,b]$, then there exists ξ between a and b such that

$$\int_a^b f(x)\,dx = f(\xi)(b-a).$$

Proof. Since f is continuous on $[a,b]$ and for $\eta := \dfrac{\int_a^b f\,dx}{b-a}$ there holds

$$\min_{[a,b]} f(x) \le \eta \le \max_{[a,b]} f(x),$$

then by the Intermediate Value Theorem for continuous functions, there exists a $\xi \in [a,b]$ such that $f(\xi) = \eta$. $\square$

Theorem. (Fundamental Theorem of Calculus, I. Derivative of an Integral) Suppose that f is continuous on $[a,b]$ and set $F(x) := \int_a^x f(y)\,dy$, then F is differentiable and $F'(x) = f(x)$ for $a < x < b$.

Proof. Notice that

$$\frac{F(x_0+h)-F(x_0)}{h} = \frac{\int_{x_0}^{x_0+h} f\,dx}{h} = f(\xi)$$

for some ξ between x_0 and x_0+h. Hence, as $h \to 0$, then $\xi = \xi_h$ converges to x_0 and so the displayed difference quotient has a limit of $f(x_0)$ as $h \to 0$. $\square$

Theorem. (Fundamental Theorem of Calculus, Part II. Integral of a Derivative) Suppose that F is function with a continuous derivative on $[a,b]$, then

$$\int_a^b F'(y)\,dy = F(x)|_{x=a}^{x=b} := F(b) - F(a)$$

.

Proof. Define $G(x) := \int_a^x F'(y)\,dy$, and set $H := F - G$. Since the derivative of H is identically zero by Part I of the Fundamental Theorem of Calculus, then the Mean Value Theorem implies that $H(b) = H(a)$. Expressing this in terms of F and G gives

$$F(b) - \int_a^b F'(y)\,dy = F(a),$$

which establishes the theorem. $\square$