Connectedness
Handout #7 – 3/20/96

Defn. A *disconnection* of a set A is two nonempty sets A_1, A_2 whose disjoint union is A and each is open relative to A. A set is said to be *connected* if it does not have any disconnections.

Example. The set $\left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ is disconnected.

Theorem. Each interval (open, closed, half-open) I is a connected set.
Proof. Let A_1, A_2 be a disconnection for I. Let $a_j \in A_j, \quad j = 1, 2$. We may assume WLOG that $a_1 < a_2$, otherwise relabel A_1 and A_2. Consider $E_1 := \{x \in A_1 | x \leq a_2\}$, then E_1 is nonempty and bounded from above. Let $a := \sup E_1$. But $a_1 \leq a \leq a_2$ implies $a \in I$ since I is an interval. First note that by the lemma to the least upper bound property either $a \in A_1$ or a is a limit point of A_1. In either case, $a \in A_1$ since A_1 is closed relative to I. Since A_1 is also open relative to the interval I, then there is an $\epsilon > 0$ so that $N_\epsilon(a) \in A_1$. But then $a + \epsilon/2 \in A_1$ and is less than a_2, which contradicts that a is the sup of E_1. $\square$

Theorem. If A is a connected set, then A is an interval.
Proof. Otherwise, there would be $a_1 < a < a_2$, with $a_j \in A$ and $a \notin A$. But then $\mathcal{O}_1 := (-\infty, a) \cap A$ and $\mathcal{O}_2 := (a, \infty) \cap A$ form a disconnection of A. $\square$

Note. Each open subset of $\mathbb{R}$ is the countable disjoint union of open intervals. This is seen by looking at open *components* (maximal connected sets) and recalling that each open interval contains a rational. Relatively (with respect to $A \subseteq \mathbb{R}$) open sets are just restrictions of these.

Theorem. The continuous image of a connected set is connected. The continuous image of $[a, b]$ is an interval $[c, d]$ where $c = \min_{a \leq x \leq b} f(x)$ and $d = \max_{a \leq x \leq b} f(x)$.

Proof. Any disconnection of the image $f([a, b])$ can be 'drawn back' to form a disconnection of $[a, b]$: if $\{\mathcal{O}_1, \mathcal{O}_2\}$ forms a disconnection for $f(I)$, then $\left\{f^{-1}(\mathcal{O}_1), f^{-1}(\mathcal{O}_2)\right\}$ forms a disconnection for $I = [a, b]$. $\square$

Corollary. (Intermediate Value Theorem) Suppose f is a real-valued function which is continuous on an interval I. If $a_1, a_2 \in I$ and y is a number between $f(a_1)$ and $f(a_2)$, then there exists a between a_1 and a_2 such that $f(a) = y$.
Proof. We may assume WLOG that $I = [a_1, a_2]$. We know that $f(I)$ is a closed

interval, say I_1. Any number y between $f(a_1)$ and $f(a_2)$, belongs to I_1 and so there is an $a \in [a_1, a_2]$ such that $f(a) = y$. $\square$

Theorem. Suppose that $f : [a, b] \to [a, b]$ is continuous, then f has a fixed point, i.e. there is an $\alpha \in [a, b]$ such that $f(\alpha) = \alpha$.
Proof. Consider the function $g(x) := x - f(x)$, then $g(a) \leq 0 \leq g(b)$. g is continuous on $[a, b]$, so by the Intermediate Value Theorem, there is an $\alpha \in [a, b]$ such that $g(\alpha) = 0$. This implies that $f(\alpha) = \alpha$. $\square$

More on Compactness

Defn. A function f is called *Lipschitz* if there is an $M > 0$ such that

$$|f(x_1) - f(x_2)| \leq M|x_1 - x_2|, \quad \text{for all } x_1, x_2 \in dom(f).$$

If $M < 1$, then f is called a *contraction.*

Theorem. Each Lipschitz function is uniformly continuous.

Theorem. Suppose that K is compact and $f : K \to K$ is a contraction, then f has a fixed point in K.
Proof. Let x_0 be an arbitrary point in K. Define inductively,

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots$$

We claim that the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent to some $\alpha \in K$. First note that for each $n \in I\!N$

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \leq M\,|x_n - x_{n-1}|.$$

Hence, by induction, for each $n \in I\!N$

$$|x_{n+1} - x_n| \leq M^n\,|x_1 - x_0|.$$

We then see that if $m > n$, then $m = n + k$ where $k \in I\!N$ and

$$\begin{aligned}
|x_{n+k} - x_n| &\leq |x_{n+k} - x_{n+k-1}| + |x_{n+k-1} - x_{n+k-2}| + \ldots + |x_{n+1} - x_n| \\
&\leq \left(M^{n+k-1} + M^{n+k-2} + \ldots + M^n\right)|x_1 - x_0| \\
&= M^n\,(1 + M + \ldots + M^{k-1})|x_1 - x_0| \\
&\leq \tfrac{|x_1 - x_0|}{1-M}\,M^n
\end{aligned}$$

and so $\{x_n\}_{n=1}^{\infty}$ is Cauchy. It must converge to some limit α which will belong to K since K is closed. But f is continuous, so $x_{n+1} = f(x_n) \to f(\alpha)$. Notice also that $x_{n+1} \to \alpha$, so α is our fixed point. $\square$

Theorem. Suppose that $f : [a, b] \to K$ is one-to-one, onto and continuous, then f^{-1} is continuous.
Proof. (#1) Suppose that $g := f^{-1}$ and $y_n \to y_0 \in K$. There exists unique $x_n \in [a, b]$ such that $f(x_n) = y_n$, or equivalently, $x_n = g(y_n)$. If $x_n \not\to x_0$, then there exists $\epsilon_0 > 0$ and a subsequence x_{n_k} such that $|x_{n_k} - x_0| \geq \epsilon_0$. This sequence in turn has a subsequence which converges in K to some $z \in K$. We

may as well assume that the subsequence is the sequence $\{x_{n_k}\}$. f is continuous so $y_{n_k} = f(x_{n_k}) \to f(z)$. But then $f(z) = y_0 = f(x_0)$. f is one-to-one, so $z = x_0$, which is a contradiction, since $|x_{n_k} - x_0| \geq \epsilon_0$. $\square$

Proof. (#2) Let $\mathcal{O} \subseteq [a,b]$ be relatively open, then $(f^{-1})^{-1}(\mathcal{O}) = f(\mathcal{O})$. Let C be the complement in $[a,b]$ of $\mathcal{O}$, then C is closed and hence compact. Therefore $f(C)$ is compact in K and conseqently it is closed. Its complement in K must then be open. That complement however is $f(\mathcal{O})$. $\square$