Math 554

Handout #5 – 2/19/96

Defn. A point x_0 is called a *limit point* of a set A if each nbhd of x_0 contains a member of A different from x_0, i.e. for each $\epsilon > 0$, $(N_\epsilon(x_0)\setminus\{x_0\}) \cap A \neq \emptyset$.

Defn. A point $x_0 \in A$ is called an *isolated point of A* if x_0 belongs to A but is not a limit point.

Theorem. A set F is closed if and only if it contains all its limit points.

Theorem. x_0 is a limit point of a set A if and only if there exists a sequence $\{x_n\} \subseteq A$ such that $x_n \to x_0$, but $x_n \neq x_0$, $(\forall n \in \mathbb{N})$.

Defn. Suppose that x_0 is a **limit point** of the domain of a function f, then f is said to have *a limit L as* $x \to x_0$ if,

$$\forall \epsilon > 0, \exists \delta > 0 \ni (x \in dom(f) \;\&\; 0 < |x - x_0| < \delta) \Longrightarrow |f(x) - L| < \epsilon.$$

In this case, we use the notation,

$$\lim_{x \to x_0} f(x) = L.$$

Theorem. Suppose that $f\colon A \to B$ is a real-valued function of a real variable, i.e. $A, B \subseteq \mathbb{R}$. If x_0 is a limit point of the domain of f, then TFAE (The Following Are Equivalent):

a.) $\lim_{x \to x_0} f(x) = L$,

b.) For every sequence $\{x_n\}$ in the domain of f, if $x_n \to x_0$, then $f(x_n) \to L$.

Defn. Suppose $A, B \subseteq \mathbb{R}$ and $f\colon A \to B$. If $x_0 \in A$, then f is said to be *continuous at* x_0 if either

1. x_0 is an isolated point of A

or

2. $\lim_{x \to x_0} f(x) = f(x_0)$.

Defn. Consider a set $B \subseteq \mathbb{R}$. A set $\tilde{O} \subseteq B$ is called *open relative to* B (or briefly, *relatively open*) if $\tilde{O} = \mathcal{O} \cap B$ for some open set $\mathcal{O} \subseteq \mathbb{R}$.

Theorem. Suppose that $f\colon A \to B$, where $A, B \subseteq \mathbb{R}$, then TFAE:

a.) f is continuous at each point of its domain,

b.) for each $x_0 \in A$ and for every $\epsilon > 0$, there is a $\delta > 0$ such that whenever $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$,

c.) if $x_n \to x_0$, then $f(x_n) \to f(x_0)$,

d.) if $f^{-1}[\mathcal{O}]$ is open for each open subset $\mathcal{O}$ of B.

Corollary. The finite sum, product, or the quotient of continuous functions is each continuous on their respective domains.

Corollary. All polynomials are continuous. Rational functions are continuous on their domains.

Theorem. The composition of continuous functions is continuous.

Examples. Each of the following are examples of continuous functions:

1. $f(x) := |x|$
2. $g(x) := \sqrt{x}$
3. $F(x) := \sqrt{\dfrac{x^2 - 2x + 5}{x^3 - 1}}$

Homework #7 (Due Friday, Feb. 23)

1. Suppose that x_0 is a limit point of the domain of both f and g have limits at x_0, then prove that
$$\lim_{x\to x_0}(f+g)(x) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x).$$

2. Suppose that f is defined by
$$f(x) := \begin{cases} 3x+2, & \text{if } -1 \leq x \\ -2x+1, & \text{if } x < -1. \end{cases}$$
Determine at each point whether or not f is continuous. Justify your answer.

3. Determine the domain of $F(x) := \sqrt{\dfrac{x^2-2x+5}{x^3-1}}$ and carefully show that F is continuous on its domain.